4.1 Composition of reloids

Definition 44 *Reloids f and g are **composable** when* $\operatorname{Dst} f = \operatorname{Src} g$.

Definition 45 *Composition of (composable) reloids is defined by the formula*

$$g \circ f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} (G \circ F) \quad | \quad F \in \operatorname{up} f, G \in \operatorname{up} g \right\}.$$

Composition of reloids is a reloid.

Theorem 42 $(h \circ g) \circ f = h \circ (g \circ f)$ *for every composable reloids f, g, h.*

Proof For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow (\forall K \in A \exists L \in B : L \subseteq K) \wedge (\forall K \in B \exists L \in A : L \subseteq K).$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B \circ A = \{L \circ K | K \in A, L \in B\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B : K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ exist $P' \in B \circ C$ such that $P' \subseteq P$; the vice versa is analogous. So $A \circ C \sim B \circ C$.

$\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ g) \circ \operatorname{up} f$, $\operatorname{up}(h \circ g) \sim (\operatorname{up} h) \circ (\operatorname{up} g)$. By proven above $\operatorname{up}((h \circ g) \circ f) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

Analogously $\operatorname{up}(h \circ (g \circ f)) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

So $\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ (g \circ f))$ what is possible only if $\operatorname{up}((h \circ g) \circ f) = \operatorname{up}(h \circ (g \circ f))$. □

Theorem 43 *For every reloid f:*

1. $f \circ f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (F \circ F) \quad | \quad F \in \operatorname{up} f \right\}$ *if* $\operatorname{Src} f = \operatorname{Dst} f$;
2. $f^{-1} \circ f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Src} f)} \left(F^{-1} \circ F \right) \quad | \quad F \in \operatorname{up} f \right\}$;
3. $f \circ f^{-1} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Dst} f)} \left(F \circ F^{-1} \right) \quad | \quad F \in \operatorname{up} f \right\}$.

Proof I will prove only (1) and (2) because (3) is analogous to (2).

1. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H \circ H \subseteq G \circ F$. To prove it take $H = F \cap G$.
2. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H^{-1} \circ H \subseteq G^{-1} \circ F$. To prove it take $H = F \cap G$. Then $H^{-1} \circ H = (F \cap G)^{-1} \circ (F \cap G) \subseteq G^{-1} \circ F$.

□