3.10 Atomic funcoids

Theorem 22 *An* $f \in \mathsf{FCD}(A;B)$ *is an atom of the lattice* $\mathsf{FCD}(A;B)$ *(for small sets* A, B*) iff it is funcoidal product of two atomic filter objects.*

Proof

- $\Rightarrow$ Let $f \in \mathsf{FCD}(A;B)$ be an atom of the lattice $\mathsf{FCD}(A;B)$. Let's get elements $a \in \operatorname{atoms} \operatorname{dom} f$ and $b \in \operatorname{atoms} \langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{F}(A)$

  $$\mathcal{X} \asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = 0^{\mathfrak{F}(B)} \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}.$$

  So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

- $\Leftarrow$ Let $a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$, $b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}$, $f \in \mathsf{FCD}(A;B)$. If $b \asymp \langle f \rangle a$ then $\neg(a \, [f] \, b)$, $f \asymp a \times^{\mathsf{FCD}} b$; if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}(A) : (\mathcal{X} \not\asymp a \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \asymp a \times^{\mathsf{FCD}} b \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atom.

□

Theorem 23 *The lattice* $\mathsf{FCD}(A;B)$ *is atomic (for every small sets* A, B*).*

Proof Let f is a non-empty funcoid from A to B. Then $\operatorname{dom} f \neq 0^{\mathfrak{F}(A)}$, thus by the theorem 47 in [15] there exists $a \in \operatorname{atoms} \operatorname{dom} f$. So $\langle f \rangle a \neq 0^{\mathfrak{F}(B)}$ thus exists $b \in \operatorname{atoms} \langle f \rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 24 *The lattice* $\mathsf{FCD}(A;B)$ *is separable (for every small sets* A, B*).*

Proof Let $f, g \in \mathsf{FCD}(A;B)$, $f \subset g$. Then exists $a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$ such that $\langle f \rangle a \subset \langle g \rangle a$. So because the lattice $\mathfrak{F}(B)$ is atomically separable then exists $b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}$ such that $\langle f \rangle a \cap b = 0^{\mathfrak{F}(B)}$ and $b \subseteq \langle g \rangle a$. For every $x \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$

$$\langle f \rangle a \cap \left\langle a \times^{\mathsf{FCD}} b \right\rangle a = \langle f \rangle a \cap b = 0^{\mathfrak{F}(B)},$$
$$x \neq a \Rightarrow \langle f \rangle x \cap \left\langle a \times^{\mathsf{FCD}} b \right\rangle x = \langle f \rangle x \cap 0^{\mathfrak{F}(B)} = 0^{\mathfrak{F}(B)}.$$

Thus $\langle f \rangle x \cap \left\langle a \times^{\mathsf{FCD}} b \right\rangle x = 0^{\mathfrak{F}(B)}$ and consequently $f \asymp a \times^{\mathsf{FCD}} b$.

$$\left\langle a \times^{\mathsf{FCD}} b \right\rangle a = b \subseteq \langle g \rangle a,$$
$$x \neq a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle x = 0^{\mathfrak{F}(B)} \subseteq \langle g \rangle x.$$

Thus $\left\langle a \times^{\mathsf{FCD}} b \right\rangle x \subseteq \langle g \rangle x$ and consequently $a \times^{\mathsf{FCD}} b \subseteq g$.

So the lattice $\mathsf{FCD}(A;B)$ is separable by the theorem 19 in [15]. □

Corollary 7 *The lattice* $\mathsf{FCD}(A;B)$ *is:*