Theorem 18 *Let A, B, C be sets, $f \in \mathsf{FCD}(A;B)$, $g \in \mathsf{FCD}(B;C)$, $h \in \mathsf{FCD}(A;C)$. Then*

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

Proof

$$\begin{aligned}
g \circ f \not\asymp h &\Leftrightarrow \\
\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : a\ [(g \circ f) \cap h]\ c &\Leftrightarrow \\
\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : (a\ [g \circ f]\ c \wedge a\ [h]\ c) &\Leftrightarrow \\
\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : (a\ [f]\ b \wedge b\ [g]\ c \wedge a\ [h]\ c) &\Leftrightarrow \\
\exists b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : \left(b\ [g]\ c \wedge b\ \left[h \circ f^{-1}\right]\ c\right) &\Leftrightarrow \\
\exists b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : b\ \left[g \cap \left(h \circ f^{-1}\right)\right]\ c &\Leftrightarrow \\
g \not\asymp h \circ f^{-1}.
\end{aligned}$$

□

3.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is funcoidal product of two filter objects:

Definition 32 ***Funcoidal product*** *of filter objects $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \in \mathsf{FCD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))$ that for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Base}(\mathcal{A}))$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Base}(\mathcal{B}))$*

$$\mathcal{X}\ \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right]\ \mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{A} \wedge \mathcal{Y} \not\asymp \mathcal{B}.$$

Proposition 17 *$\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a funcoid and*

$$\left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A}; \\ 0^{\mathfrak{F}(\operatorname{Base}(\mathcal{B}))} & \text{if } \mathcal{X} \asymp \mathcal{A}. \end{cases}$$

Proof Obvious. □

Obvious 12. $\uparrow^{\mathsf{FCD}(U;V)} (A \times B) = \uparrow^U A \times^{\mathsf{FCD}} \uparrow^V B$ for sets $A \subseteq U$ and $B \subseteq V$ (for some small sets U and V).

Proposition 18 *$f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ for every $f \in \mathsf{FCD}(A;B)$ and $\mathcal{A} \in \mathfrak{F}(A)$, $\mathcal{B} \in \mathfrak{F}(B)$.*

Proof If $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \subseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{B}$. If $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ then

$$\forall \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B) : (\mathcal{X}\ [f]\ \mathcal{Y} \Rightarrow \mathcal{X} \cap \mathcal{A} \neq 0^{\mathfrak{F}(A)} \wedge \mathcal{Y} \cap \mathcal{B} \neq 0^{\mathfrak{F}(B)});$$

consequently $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids: