Proof. $\operatorname{Join}(P, h)(f)(x) = (h \circ f)(x) = h f x \sqsupseteq h' f x = (h' \circ f)(x) = \operatorname{Join}(P, h')(f)(x)$. ☐

Lemma 26. If $g: Q \rightarrow R$ and $h: R \rightarrow S$ are finite joins preserving, then the composition $\operatorname{Join}(P; h) \circ \operatorname{Join}(P; g)$ is equal to $\operatorname{Join}(P; h \circ g)$. Also $\operatorname{Join}(P; \mathrm{id}_Q)$ for identity map id_Q on Q is the identity map $\mathrm{id}_{\operatorname{Join}(P;Q)}$ on $\operatorname{Join}(P; Q)$.

Proof. $\operatorname{Join}(P; h) \operatorname{Join}(P; g)\, f = \operatorname{Join}(P; h)(g \circ f) = h \circ g \circ f = \operatorname{Join}(P; h \circ g) f$.

$\operatorname{Join}(P; \mathrm{id}_Q)\, f = \mathrm{id}_Q \circ f = f$. ☐

Corollary 27. If Q is a join-semilattice and $F: Q \rightarrow Q$ is a co-nucleus, then for any join-semilattice P we have that $\operatorname{Join}(P; F): \operatorname{Join}(P; Q) \rightarrow \operatorname{Join}(P; Q)$ is also a co-nucleus.

Proof. From $\mathrm{id}_Q \sqsupseteq F$ (co-nucleus axiom 1) we have $\operatorname{Join}(P; \mathrm{id}_Q) \sqsupseteq \operatorname{Join}(P; F)$ and since by the last lemma the left side is the identity on $\operatorname{Join}(P; Q)$, we see that $\operatorname{Join}(P; F)$ also satisfies co-nucleus axiom 1.

$\operatorname{Join}(P;\ F) \circ \operatorname{Join}(P;\ F) = \operatorname{Join}(P;\ F \circ F)$ by the same lemma and thus $\operatorname{Join}(P;\ F) \circ \operatorname{Join}(P;\ F) = \operatorname{Join}(P; F)$ by the second co-nucleus axiom for F, showing that $\operatorname{Join}(P; F)$ satisfies the second co-nucleus axiom.

By an other lemma, we have that $\operatorname{Join}(P; F)$ preserves finite joins, given that F preserves finite joins, which is the third co-nucleus axiom. ☐

Lemma 28. $\operatorname{Fix}(\operatorname{Join}(P;\ F)) = \operatorname{Join}(P;\ \operatorname{Fix}(F))$ for every join-semilattices P, Q and a join preserving function $F: Q \rightarrow Q$.

Proof. $a \in \operatorname{Fix}(\operatorname{Join}(P; F)) \Leftrightarrow a \in F^P \wedge F \circ a = a \Leftrightarrow a \in F^P \wedge \forall x \in P: F(a(x)) = a(x)$.

$a \in \operatorname{Join}(P; \operatorname{Fix}(F)) \Leftrightarrow a \in \operatorname{Fix}(F)^P \Leftrightarrow a \in F^P \wedge \forall x \in P: F(a(x)) = a(x)$.

Thus $\operatorname{Fix}(\operatorname{Join}(P; F)) = \operatorname{Join}(P; \operatorname{Fix}(F))$. That the order of the left and right sides of the equality agrees is obvious. ☐

Definition 29. $\mathbf{Pos}(\mathfrak{A}; \mathfrak{B})$ is the pointwise ordered poset of monotone maps from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

Lemma 30. If Q, R are join-semilattices and P is a poset, then $\mathbf{Pos}(P; R)$ is a join-semilattice and $\mathbf{Pos}(P; \operatorname{Join}(Q; R))$ is isomorphic to $\operatorname{Join}(Q; \mathbf{Pos}(P; R))$. If R is a co-frame, then also $\mathbf{Pos}(P; R)$ is a co-frame.

Proof. Let $f, g \in \mathbf{Pos}(P; R)$. Then $\lambda x \in P: (f x \sqcup g x)$ is obviously monotone and then it is evident that $f \sqcup^{\mathbf{Pos}(P;R)} g = \lambda x \in P: (f x \sqcup g x)$. $\lambda x \in P: \perp^R$ is also obviously monotone and it is evident that $\perp^{\mathbf{Pos}(P;R)} = \lambda x \in P: \perp^R$.

Obviously both $\mathbf{Pos}(P; \operatorname{Join}(Q; R))$ and $\operatorname{Join}(Q; \mathbf{Pos}(P; R))$ are sets of order preserving maps.

Let f be a monotone map.

$f \in \mathbf{Pos}(P; \operatorname{Join}(Q; R))$ iff $f \in \operatorname{Join}(Q; R)^P$ iff $f \in \{g \in R^Q \mid g \text{ preserves finite joins}\}^P$ iff $f \in (R^Q)^P$ and every $g = f(x)$ (for $x \in P$) preserving finite joins. This is bijectively equivalent ($f \mapsto f'$) to $f' \in (R^P)^Q$ preserving finite joins.

$f' \in \operatorname{Join}(Q; \mathbf{Pos}(P; R))$ iff f' preserves finite joins and $f' \in \mathbf{Pos}(P; R)^Q$ iff f' preserves finite joins and $f' \in \{g \in (R^P)^Q \mid g(x) \text{ is monotone}\}$ iff f' preserves finite joins and $f' \in (R^P)^Q$.

So we have proved that $f \mapsto f'$ is a bijection between $\mathbf{Pos}(P; \operatorname{Join}(Q; R))$ and $\operatorname{Join}(Q; \mathbf{Pos}(P; R))$. That it preserves order is obvious.