$\Leftarrow$. Suppose $\bigsqcap$: $\mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins. Let $b \in \mathfrak{A}$, $S \in \mathscr{P}\mathfrak{A}$. Let D be the smallest upper set containing S (so $D = \bigcup \langle\uparrow\rangle^* S$). We have $\bigsqcap D = \bigcup S$. So

$$\begin{aligned}
b \sqcup \bigsqcap S &= \\
\bigsqcap \uparrow b \sqcup \bigcup \bigsqcap \langle\uparrow\rangle^* S &= \\
\bigsqcap \uparrow b \sqcup \bigsqcap \bigcup \langle\uparrow\rangle^* S &= \text{(since } \textstyle\bigsqcap \text{ preserves finite joins)} \\
\bigsqcap \left(\uparrow b \sqcup \bigcup \langle\uparrow\rangle^* S\right) &= \\
\bigcup \left(\uparrow b \cap \bigcup \langle\uparrow\rangle^* S\right) &= \\
\bigsqcap \bigcup_{a \in S} (\uparrow b \cap \uparrow a) &= \\
\bigsqcap \bigcup_{a \in S} \uparrow(b \sqcup a) &= \text{(since } \textstyle\bigsqcap \text{ preserves all meets)} \\
\bigcup_{a \in S} \bigsqcap \uparrow(b \sqcup a) &= \\
\bigcup_{a \in S} (b \sqcup a) &= \\
\bigsqcap_{a \in S} (b \sqcup a). &
\end{aligned}$$

□

Corollary 20. If $\mathfrak{A}$ is a co-frame, then the composition $F = \uparrow \circ \bigsqcap$: $\mathrm{Up}(\mathfrak{A}) \to \mathrm{Up}(\mathfrak{A})$ is a co-nucleus. The embedding $\uparrow$: $\mathfrak{A} \to \mathrm{Up}(\mathfrak{A})$ is an isomorphism of $\mathfrak{A}$ onto the co-frame $\mathrm{Fix}(F)$.

Proof. $D \sqsupseteq F(D)$ follows from theorem 16.

We have $F(F(D)) = F(D)$ for all $D \in \mathrm{Up}(\mathfrak{A})$ since $F(F(D)) = \uparrow\bigsqcap \uparrow\bigsqcap D =$ (because $\bigsqcap \uparrow s = s$ for any s)$= \uparrow\bigsqcap D = F(D)$.

And since both $\bigsqcap$: $\mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ and $\uparrow$ preserve finite joins, F preserves finite joins. Thus F is a co-nucleus.

Finally, we have $a \sqsupseteq a'$ if and only if $\uparrow a \subseteq \uparrow a'$, so that $\uparrow$: $\mathfrak{A} \to \mathrm{Up}(\mathfrak{A})$ maps $\mathfrak{A}$ isomorphically onto its image $\langle\uparrow\rangle^*\mathfrak{A}$. This image is $\mathrm{Fix}(F)$ because if D is any fixed point (i.e. if $D = \uparrow\bigsqcap D$), then D clearly belongs to $\langle\uparrow\rangle^*\mathfrak{A}$; and conversely $\uparrow a$ is always a fixed point of $F = \uparrow \circ \bigsqcap$ since $F(\uparrow a) = \uparrow\bigsqcap \uparrow a = \uparrow a$. □

Definition 21. If $\mathfrak{A}$, $\mathfrak{B}$ are two join-semilattices, then $\mathrm{Join}(\mathfrak{A}; \mathfrak{B})$ is the (ordered pointwise) set of finite joins preserving maps $\mathfrak{A} \to \mathfrak{B}$.

Obvious 22. $\mathrm{Join}(\mathfrak{A}; \mathfrak{B})$ is a join-semilattice, where $f \sqcup g$ is given by the formula $(f \sqcup g)(p) = f(p) \sqcup g(p)$, $\bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})}$ is given by the formula $\bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})}(p) = \bot^{\mathfrak{B}}$.

Definition 23. Let h: $Q \to R$ be a finite joins preserving map. Then by definition $\mathrm{Join}(P, h)$: $\mathrm{Join}(P; Q) \to \mathrm{Join}(P; R)$ takes $f \in \mathrm{Join}(P; Q)$ into the composition $h \circ f \in \mathrm{Join}(P; R)$.

Lemma 24. Above defined $\mathrm{Join}(P, h)$ is a finite joins preserving map.

Proof. $(h \circ (f \sqcup f'))x = h(f \sqcup f')x = h(fx \sqcup f'\,x) = hfx \sqcup hf'\,x = (h \circ f)x \sqcup (h \circ f')x = ((h \circ f) \sqcup (h \circ f'))x$. Thus $h \circ (f \sqcup f') = (h \circ f) \sqcup (h \circ f')$.

$\left(h \circ \bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})}\right)x = h\bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})}x = h\bot^{\mathfrak{B}} = \bot^{\mathfrak{A}}$. □

Proposition 25. If h, h': $Q \to R$ are finite join preserving maps and $h \sqsupseteq h'$, then $\mathrm{Join}(P, h) \sqsupseteq \mathrm{Join}(P, h')$.