Take any subset S of $\mathfrak{A}$. Let D be the smallest upper set containing S. (It exists because $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary joins.) This is

$$D = \{x \in \mathfrak{A} \mid \exists s \in S : x \sqsupseteq s\}.$$

Any lower bound of D is clearly an upper bound of S since $D \supseteq S$. Conversely any lower bound of S is a lower bound of D. Thus S and D have the same set of lower bounds, hence have the same greatest lower bound. □

Proposition 17. [TODO: Move it above in the book.] For any poset $\mathfrak{A}$ the following are mutually reverse order isomorphisms between upper sets F (ordered reverse to set-theoretic inclusion) on $\mathfrak{A}$ and order homomorphisms $\varphi : \mathfrak{A}^{\mathrm{op}} \to 2$ (here 2 is the partially ordered set of two elements: 0 and 1 where $0 \sqsubseteq 1$), defined by the formulas

1. $\varphi(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ for every $a \in \mathfrak{A}$;

2. $F = \varphi^{-1}(1)$.

Proof. Let $X \in \varphi^{-1}(1)$ and $Y \sqsupseteq X$. Then $\varphi(X) = 1$ and thus $\varphi(Y) = 1$. Thus $\varphi^{-1}(1)$ is a upper set.

It is easy to show that φ defined by the formula (1) is an order homomorphism $\mathfrak{A}^{\mathrm{op}} \to 2$ whenever F is a upper set.

Finally we need to prove that they are mutually inverse. Really: Let φ be defined by the formula (1). Then take $F' = \varphi^{-1}(1)$ and define $\varphi'(a)$ by the formula (1). We have

$$\varphi'(a) = \begin{cases} 1 & \text{if } a \in \varphi^{-1}(1) \\ 0 & \text{if } a \notin \varphi^{-1}(1) \end{cases} = \begin{cases} 1 & \text{if } \varphi(a) = 1 \\ 0 & \text{if } \varphi(a) \neq 1 \end{cases} = \varphi(a).$$

Let now F be defined by the formula (2). Then take $\varphi'(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ as defined by the formula (1) and define $F' = \varphi'^{-1}(1)$. Then

$$F' = \varphi'^{-1}(1) = F.$$ □

Lemma 18. For a complete lattice $\mathfrak{A}$, the map $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves arbitrary meets.

Proof. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$. We have $\bigsqcap S \in \operatorname{Up}(\mathfrak{A})$.

$\bigsqcap \bigsqcap S = \bigsqcap \bigsqcap_{X \in S} X = \bigsqcap_{X \in S} \bigsqcap X$ is what we needed to prove. □

Lemma 19. A complete lattice $\mathfrak{A}$ is a co-frame iff $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins.

Proof.

⇒. Let $\mathfrak{A}$ be a co-frame. Let $D, D' \in \operatorname{Up}(\mathfrak{A})$. Obviously $\bigsqcap (D \sqcup D') \sqsupseteq \bigsqcap D$ and $\bigsqcap (D \sqcup D') \sqsupseteq \bigsqcap D'$, so $\bigsqcap (D \sqcup D') \sqsupseteq \bigsqcap D \sqcup \bigsqcap D'$.

Also $\bigsqcap D \sqcup \bigsqcap D' = \bigcup D \sqcup \bigcup D' =$ (because $\mathfrak{A}$ is a co-frame)$= \bigcup \{d \sqcup d' \mid d \in D, d' \in D'\}$. Obviously $d \sqcup d' \in D \cap D'$, thus $\bigsqcap D \sqcup \bigsqcap D' \subseteq \bigcup (D \cap D') = \bigsqcap (D \cap D')$ that is $\bigsqcap D \sqcup \bigsqcap D' \sqsupseteq \bigsqcap (D \cap D')$. So $\bigsqcap (D \sqcup D') = \bigsqcap D \sqcup \bigsqcap D'$ that is $\bigsqcap : \operatorname{Up}(A) \to A$ preserves binary joins.

It preserves nullary joins since $\bigsqcap^{\operatorname{Up}(\mathfrak{A})} \bot^{\operatorname{Up}(\mathfrak{A})} = \bigsqcap^{\operatorname{Up}(\mathfrak{A})} \mathfrak{A} = \bot^{\mathfrak{A}}$.