Obvious 12. $\operatorname{Fix}(F)$ with induced order is a complete lattice.

Lemma 13. If F is a co-nucleus on a co-frame $\mathfrak{A}$, then the poset $\operatorname{Fix}(F)$ of fixed points of F, with order inherited from $\mathfrak{A}$, is also a co-frame.

Proof. Let $b \in \operatorname{Fix}(F)$, $S \in \mathscr{P}\operatorname{Fix}(F)$. Then

$$\begin{aligned} b \sqcup^{\operatorname{Fix}(F)} \bigsqcap^{\operatorname{Fix}(F)} S &= \\ b \sqcup^{\operatorname{Fix}(F)} F\left(\bigsqcap S\right) &= \\ F(b) \sqcup F\left(\bigsqcap S\right) &= \\ F\left(b \sqcup \bigsqcap S\right) &= \\ F\left(\bigsqcap \langle b \sqcup \rangle^{*} S\right) &= \\ \bigsqcap^{\operatorname{Fix}(F)} \langle b \sqcup \rangle^{*} S &= \\ \bigsqcap^{\operatorname{Fix}(F)} \left\langle b \sqcup^{\operatorname{Fix}(F)} \right\rangle^{*} S. & \end{aligned}$$

□

Definition 14. *Upper set* is a set X on a poset $\mathfrak{A}$ such that $x \in X \wedge y \sqsupseteq x \Rightarrow y \in X$ for every $y \in \mathfrak{A}$.

Denote $\operatorname{Up}(\mathfrak{A})$ the set of upper sets on $\mathfrak{A}$ ordered *reverse* to set theoretic inclusion. [TODO: move it above in the book]

Lemma 15. The set $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary meets and joins.

Proof. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$.

Let $X \in \bigcup S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then there is $P \in S$ such that $X \in P$ and thus $Y \in P$ and so $Y \in \bigcup S$. So $\bigcup S \in \operatorname{Up}(\mathfrak{A})$.

Let now $X \in \bigcap S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then $\forall T \in S: X \in T$ and so $\forall T \in S: Y \in T$, thus $Y \in \bigcap S$. So $\bigcap S \in \operatorname{Up}(\mathfrak{A})$. □

Theorem 16. A poset $\mathfrak{A}$ is a complete lattice iff there is a antitone map $s: \operatorname{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ such that [TODO: define *antitone*.]

1. $s(\uparrow p) = p$ for every $p \in \mathfrak{A}$;
2. $D \subseteq \uparrow s(D)$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

Moreover, in this case $s(D) = \bigsqcap D$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

Proof.

⇒. Take $s(D) = \bigsqcap D$.

⇐. $\forall x \in D: x \sqsupseteq s(D)$ from the second formula.

Let $\forall x \in D: y \sqsubseteq x$. Then $x \in \uparrow y$, $D \subseteq \uparrow y$; because s is an antitone map, thus follows $s(D) \sqsupseteq s(\uparrow y) = y$. So $\forall x \in D: y \sqsubseteq s(D)$.

That s is the meet follows from the definition of meets.

It remains to prove that $\mathfrak{A}$ is a complete lattice.