In this chapter the term *join-semilattice* means join-semilattice with least element $\bot$.

Definition 1. A *co-frame* is the same as a complete co-brouwerian lattice. [TODO: move it above in the book and use it when appropriate]

Definition 2. It is said that a function f from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ *preserves finite joins*, when for every finite set $S \in \mathscr{P}\mathfrak{A}$ such that $\bigsqcup^{\mathfrak{A}} S$ exists we have $\bigsqcup^{\mathfrak{B}} \langle f \rangle^* S = f \bigsqcup^{\mathfrak{A}} S$.

Obvious 3. A function between join-semilattices preserves finite joins iff it preserves binary joins ($f(x \sqcup y) = fx \sqcup fy$) and nullary joins ($f(\bot^{\mathfrak{A}}) = \bot^{\mathfrak{B}}$).

Definition 4. A *fixed point* of a function F is such x that $F(x) = x$. We will denote $\operatorname{Fix}(F)$ the set of all fixed points of a function F.

Remark 5. This is based on a Todd Trimble's proof. A shorter but less elementary proof (also by Todd Trimble) is available at http://ncatlab.org/toddtrimble/published/topogeny

Definition 6. Let $\mathfrak{A}$ be a join-semilattice. A *co-nucleus* is a function $F: \mathfrak{A} \rightarrow \mathfrak{A}$ such that for every $p, q \in \mathfrak{A}$ we have:

1. $F(p) \sqsubseteq p$;
2. $F(F(p)) = F(p)$;
3. $F(p \sqcup q) = F(p) \sqcup F(q)$.

Proposition 7. Every co-nucleus is a monotone function.

Proof. It follows from $F(p \sqcup q) = F(p) \sqcup F(q)$. □

Lemma 8. $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ for every $S \in \mathscr{P}\operatorname{Fix}(F)$ for every co-nucleus F.

Proof. Obviously $\bigsqcup S \sqsupseteq x$ for every $x \in S$.

Suppose $z \sqsupseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsupseteq \bigsqcup S$.

$F(\bigsqcup S) \sqsupseteq F(x)$ for every $x \in S$. Thus $F(\bigsqcup S) \sqsupseteq \bigsqcup_{x \in S} F(x) = \bigsqcup S$. But $F(\bigsqcup S) \sqsubseteq \bigsqcup S$. Thus $F(\bigsqcup S) = \bigsqcup S$ that is $\bigsqcup S \in \operatorname{Fix}(F)$.

So $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ by the definition of join. □

Corollary 9. $\bigsqcup^{\operatorname{Fix}(F)} S$ is defined for every $x, y \in \operatorname{Fix}(F)$.

Lemma 10. $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ for every $S \in \mathscr{P}\operatorname{Fix}(F)$ for every co-nucleus F.

Proof. Obviously $F(\bigsqcap S) \sqsubseteq x$ for every $x \in S$.

Suppose $z \sqsubseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsubseteq \bigsqcap S$ and thus $z \sqsubseteq F(\bigsqcap S)$.

So $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ by the definition of meet. □

Corollary 11. $\bigsqcap^{\operatorname{Fix}(F)} S$ is defined for every $x, y \in \operatorname{Fix}(F)$.