Proposition 9. If $X \in Q$ then $X = \bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X)$.

Proof. $\bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X) \subseteq X$ is obvious.

Let $x \in X$. Then there is $Y \in \mathfrak{R}(Q)$ such that $x \in Y$. We have $Y \subseteq X$ that is $Y \in \mathscr{P}X$ by a proposition above. So $x \in Y$ where $Y \in \mathfrak{R}(Q) \cap \mathscr{P}X$ and thus $x \in \bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X)$. We have $X \subseteq \bigcup (\mathfrak{R}(Q) \cap \mathscr{P}X)$. □

3 Finite unions of Cartesian products

Let A, B be sets.

I will denote $\overline{X} = A \setminus X$.

Let denote $\Gamma(A; B)$ the set of all finite unions $X_0 \times Y_0 \cup ... \cup X_{n-1} \times Y_{n-1}$ of Cartesian products, where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, ..., n-1$.

Proposition 10. The following sets are pairwise equal:

1. $\Gamma(A; B)$;
2. the set of all sets of the form $\bigcup_{X \in S} (X \times Y_X)$ where S are finite collections on A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
3. the set of all sets of the form $\bigcup_{X \in S} (X \times Y_X)$ where S are finite partitions of A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
4. the set of all finite unions $\bigcup_{(X;Y) \in \sigma} (X \times Y)$ where σ is a relation between a partition of A and a partition of B (that is $\operatorname{dom} \sigma$ is a partition of A and $\operatorname{im} \sigma$ is a partition of B).
5. the set of all finite intersections $\bigcap_{i=0,\ldots,n-1} (X_i \times Y_i \cup \overline{X_i} \times B)$ where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, ..., n-1$.

Proof.

(1)⊇(2), (2)⊇(3). Obvious.

(1)⊆(2). Let $Q \in \Gamma(A; B)$. Then $Q = X_0 \times Y_0 \cup ... \cup X_{n-1} \times Y_{n-1}$. Denote $S = \{X_0, ..., X_{n-1}\}$. We have $Q = \bigcup_{X' \in S} (X' \times \bigcup \{Y_i \mid X_i = X'\}) \in (2)$.

(2)⊆(3). Let $Q = \bigcup_{X \in S} (X \times Y_X)$ where S is a finite collection on A and $Y_X \in \mathscr{P}B$ for every $X \in S$. Let

$$P = \bigcup_{X' \in \mathfrak{R}(S)} \left(X' \times \bigcup \{Y_X \mid X \in S \wedge X' \subseteq X\}\right)$$

To finish the proof let's show $P = Q$.

$\langle P \rangle^* \{x\} = \bigcup \{Y_X \mid X \in S \wedge X' \subseteq X\}$ where $x \in X'$.

Thus $\langle P \rangle^* \{x\} = \bigcup \{Y_X \mid X \in S \wedge x \in X\} = \langle Q \rangle^* \{x\}$. So $P = Q$.

(4)⊆(3). $\bigcup_{(X;Y) \in \sigma} (X \times Y) = \bigcup_{X \in \operatorname{dom} \sigma} (X \times \bigcup \{Y \in \mathscr{P}B \mid (X; Y) \in \sigma\}) \in (3)$.

(3)⊆(4). $\bigcup_{X \in S} (X \times Y_X) = \bigcup_{X \in S} (X \times \bigcup (\mathfrak{R}(\{Y_X \mid X \in S\}) \cap \mathscr{P}Y_X)) = \bigcup_{X \in S} (X \times \bigcup \{Y' \in \mathfrak{R}(\{Y_X \mid X \in S\}) \mid Y' \subseteq Y_X\}) = \bigcup_{X \in S} (X \times \bigcup \{Y' \in \mathfrak{R}(\{Y_X \mid X \in S\}) \mid (X; Y') \in \sigma\}) = \bigcup_{(X;Y) \in \sigma} (X \times Y)$ where σ is a relation between S and $\mathfrak{R}(\{Y_X \mid X \in S\})$, and $(X; Y') \in \sigma \Leftrightarrow Y' \subseteq Y_X$.

(5)⊆(3). Obvious.

(3)⊆(5). Let $Q = \bigcup_{X \in S} (X \times Y_X) = \bigcup_{i=0,\ldots,n-1} (X_i \times Y_i)$ for a partition $S = \{X_0, ..., X_{n-1}\}$ of A. Then $Q = \bigcap_{i=0,\ldots,n-1} (X_i \times Y_i \cup \overline{X_i} \times B)$. □

Exercise 1. Formulate the duals of these sets.

Proposition 11. $\Gamma(A; B)$ is a boolean lattice, a sublattice of the lattice $\mathscr{P}(A \times B)$.

Proof. That it's a sublattice is obvious. That it has complement, is also obvious. Distributivity follows from distributivity of $\mathscr{P}(A \times B)$. □