Funcoids are Filters

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1 Draft status

This is a rough draft.

In this article notations are used accordingly:

http://www.mathematics21.org/binaries/rewrite-plan.pdf

Particularly $\langle f\rangle^\ast X \overset{\text{def}}{=} \{ y \mid x \in X \wedge x\, f\, y\}$ for a binary relation f and a set X.

The motto of this article is: "Funcoids are filters on a lattice."

2 Rearrangement of collections of sets

Let Q is a set of sets.

Let $\equiv$ be the relation on $\bigcup Q$ defined by the formula

$$a \equiv b \Leftrightarrow \forall X \in Q: (a \in X \Leftrightarrow b \in X).$$

[TODO: Generalize it by the formula $a \equiv b \Leftrightarrow \forall X \in Q: (a \in \operatorname{atoms} X \Leftrightarrow b \in \operatorname{atoms} X)$.]

[TODO: Reloids $\mathsf{RLD}(\mathfrak{A}; \mathfrak{B})$ between posets $\mathfrak{A}$ and $\mathfrak{B}$ is $\mathfrak{F}(\operatorname{atoms}^{\mathfrak{A}} \times \operatorname{atoms}^{\mathfrak{B}})$?]

Proposition 1. $\equiv$ is an equivalence relation on $\bigcup Q$.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $a \equiv b \wedge b \equiv c$. Then $a \in X \Leftrightarrow b \in X \Leftrightarrow c \in X$ for every $X \in Q$. Thus $a \equiv c$. □

Definition 2. *Rearrangement* $\mathfrak{R}(Q)$ of Q is the set of equivalence classes of $\bigcup Q$ for $\equiv$.

Obvious 3. $\bigcup \mathfrak{R}(Q) = \bigcup Q$.

Obvious 4. $\emptyset \notin \mathfrak{R}(Q)$.

Lemma 5. $\operatorname{card} \mathfrak{R}(Q) \leq 2^{\operatorname{card} Q}$.

Proof. Having an equivalence class C, we can find the set $f \in \mathscr{P} Q$ of all $X \in Q$ such that $a \in X$ for all $a \in C$. $b \equiv a \Leftrightarrow \forall X \in Q: (a \in X \Leftrightarrow b \in X) \Leftrightarrow \forall X \in Q: (X \in f \Leftrightarrow b \in X)$. So $C = \{ b \in \bigcup Q \mid b \equiv a\}$ can be restored knowing f. Consequently there are no more than $\operatorname{card} \mathscr{P} Q = 2^{\operatorname{card} Q}$ classes. □

Corollary 6. If Q is finite, then $\mathfrak{R}(Q)$ is finite.

Proposition 7. If $X \in Q$, $Y \in \mathfrak{R}(Q)$ then $X \cap Y \neq \emptyset \Leftrightarrow Y \subseteq X$.

Proof. Let $X \cap Y \neq \emptyset$ and $x \in X \cap Y$. Then $y \in Y \Leftrightarrow x \equiv y \Leftrightarrow \forall X' \in Q: (x \in X' \Leftrightarrow y \in X') \Rightarrow (x \in X \Leftrightarrow y \in X) \Leftrightarrow y \in X$ for every y. Thus $Y \subseteq X$.

$Y \subseteq X \Rightarrow X \cap Y \neq \emptyset$ because $Y \neq \emptyset$. □

Proposition 8. If $\emptyset \neq X \in Q$ then there exists $Y \in \mathfrak{R}(Q)$ such that $Y \subseteq X \wedge X \cap Y \neq \emptyset$.

Proof. Let $a \in X$. Then $[a] = \{ b \in \bigcup Q \mid \forall X' \in Q: (a \in X' \Leftrightarrow b \in X')\} = \{ b \in \bigcup Q \mid \forall X' \in Q: b \in X'\} \subseteq \{ b \in \bigcup Q \mid b \in X\} = X$. But $[a] \in \mathfrak{R}(Q)$.

$X \cap Y \neq \emptyset$ follows from $Y \subseteq X$ by the previous proposition. □