Example 2 There exists $A \in \mathcal{P}U$ such that $\bigcap^{\mathfrak{F}} A \neq \bigcap^{\mathcal{P}U} A$ for some set U.

Proof $\bigcap^{\mathcal{P}\mathbb{R}} \{(-a; a) \mid a \in \mathbb{R}, a > 0\} = \{0\} \neq \Delta$. □

Example 3 There exists a set U and there are a f.o. a and a set S of f.o. on the lattice $\mathcal{P}U$ such that $a \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} S \neq \bigcup^{\mathfrak{F}} \langle a \cap^{\mathfrak{F}} \rangle S$.

Proof Let $a = \Delta$ and $S = \{(\varepsilon; +\infty) \mid \varepsilon > 0\}$. Then $a \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} S = \Delta \cap^{\mathfrak{F}} (0; +\infty) \neq \emptyset$ while $\bigcup^{\mathfrak{F}} \langle a \cap^{\mathfrak{F}} \rangle S = \bigcup^{\mathfrak{F}} \{\emptyset\} = \emptyset$. □

Example 4 There are thornings which are not weak partitions.

Proof $\{\Delta + a \mid a \in \mathbb{R}\}$ is a thorning but not weak partition of the real line. □

Lemma 7 *Let $\mathfrak{F}$ be the set of f.o. on a set U. Then $X \cap^{\mathfrak{F}} \Omega \subseteq Y \cap^{\mathfrak{F}} \Omega$ iff $X \setminus Y$ is a finite set, for every sets $X, Y \in \mathcal{P}U$.*

Proof $X \cap^{\mathfrak{F}} \Omega \subseteq Y \cap^{\mathfrak{F}} \Omega \Leftrightarrow \{X \cap K_X \mid K_X \in \operatorname{up} \Omega\} \supseteq \{Y \cap K_Y \mid K_Y \in \operatorname{up} \Omega\} \Leftrightarrow \forall K_Y \in \operatorname{up} \Omega \exists K_X \in \operatorname{up} \Omega : Y \cap K_Y = X \cap K_X \Leftrightarrow \forall L_Y \in M \exists L_X \in M : Y \setminus L_Y = X \setminus L_X \Leftrightarrow \forall L_Y \in M : X \setminus (Y \setminus L_Y) \in M \Leftrightarrow X \setminus Y \in M$ where M is the set of finite subsets of U. □

Example 5 There exists a filter object $\mathcal{A}$ on a set U such that $(\mathcal{P}U)/\sim$ and $Z(D\mathcal{A})$ are not complete lattices.

Proof Due isomorphism it's enough to prove for $(\mathcal{P}U)/\sim$.

Let's take $U = \mathbb{N}$ and $\mathcal{A} = \Omega$ be the Frechet filter object on $\mathbb{N}$.

Partition $\mathbb{N}$ into infinitely many infinite sets $A_0, A_1, \ldots$. To withhold our example we will prove that the set $\{[A_0], [A_1], \ldots\}$ has no supremum in $(\mathcal{P}U)/\sim$.

Let $[X]$ be an upper bound of $[A_0], [A_1], \ldots$ that is $\forall i \in \mathbb{N} : X \cap \Omega \supseteq A_i \cap \Omega$ that is $A_i \setminus X$ is finite. Consequently X is infinite. So $X \cap A_i \neq \emptyset$.

Choose for every $i \in \mathbb{N}$ some $z_i \in X \cap A_i$. Then $\{z_0, z_1, \ldots\}$ is an infinite subset of X (take in account that $z_i \neq z_j$ for $i \neq j$). Let $Y = X \setminus \{z_0, z_1, \ldots\}$. Then $Y \cap^{\mathfrak{F}} \Omega \supseteq A_i \cap^{\mathfrak{F}} \Omega$ because $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$ which is finite because $A_i \setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0], [A_1], \ldots\}$.

Suppose $Y \cap^{\mathfrak{F}} \Omega = X \cap^{\mathfrak{F}} \Omega$. Then $Y \setminus X$ is finite what is not true. So $Y \cap^{\mathfrak{F}} \Omega \subset X \cap^{\mathfrak{F}} \Omega$ that is $[Y]$ is below $[X]$. □

Appendix A.1. Weak and strong partition

Definition 78 *A family S of subsets of a countable set is **independent** iff the intersection of any finitely many members of S and the complements of any other finitely many members of S is infinite.*