19. Open problems

In this section I will formulate some conjectures about lattices of filter objects on a set. If a conjecture comes true, it may be generalized for more general lattices (such as, for example, lattices of filters on arbitrary lattices). I deem that the main challenge is to prove the special case about lattices of filter objects on a set, and generalizing the conjectures is expected to be a simple task.

19.1. Partitioning

Consider the complete lattice $[S]$ generated by the set S where S is a strong partition of some element a.

Conjecture 1 $[S] = \left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$*, where* $[S]$ *is the complete lattice generated by a strong partition* S *of an element of a lattice* $\mathfrak{F}$ *of filter objects on a set.*

Proposition 39 *Provided that the last conjecture is true, we have that* $[S]$ *is a complete atomic boolean lattice with the set of its atoms being* S*.*

Remark 13 Consequently S is atomistic, completely distributive and isomorphic to a power set algebra (see [13]).

Proof Completeness of $[S]$ is obvious. Let $A \in [S]$. Then exists $X \in \mathcal{P}S$ such that $A = \bigcup^{\mathfrak{F}} X$. Let $B = \bigcup^{\mathfrak{F}} (S \setminus X)$. Then $B \in [S]$ and $A \cap B = 0$. $A \cup B = \bigcup^{\mathfrak{F}} S$ is the biggest element of $[S]$. So we have proved that $[S]$ is a boolean lattice.

Now let prove that $[S]$ is atomic with the set of atoms being S. Let $z \in S$ and $A \in [S]$. If $A \neq z$ then either $A = 0$ or $x \in X$ where $A = \bigcup^{\mathfrak{F}} X$, $X \in \mathcal{P}S$ and $x \neq z$. Because S is a partition, $\bigcup^{\mathfrak{F}} (X \setminus \{z\}) \cap^{\mathfrak{F}} z = 0$ and $\bigcup^{\mathfrak{F}} (X \setminus \{z\}) \neq 0$. So $A = \bigcup^{\mathfrak{F}} X = \bigcup^{\mathfrak{F}} (X \setminus \{z\}) \cup^{\mathfrak{F}} z \nsubseteq z$.

Finally we will prove that elements of $[S] \setminus S$ are not atoms. Let $A \in [S] \setminus S$ and $A \neq 0$. Then $A \supseteq x \cup^{\mathfrak{F}} y$ where $x, y \in S$ and $x \neq y$. If A is an atom then $A = x = y$ what is impossible. □

Proposition 40 *The conjecture about the value of* $[S]$ *is equivalent to closedness of* $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ *under arbitrary meets and joins.*

Proof If $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\} = [S]$ then trivially $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ is closed under arbitrary meets and joins.

If $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ is closed under arbitrary meets and joins, then it is the complete lattice generated by the set S because it cannot be smaller than the set of all suprema of subsets of S. □

That $\left\{ \bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S \right\}$ is closed under arbitrary joins is trivial. I have not succeeded to prove that it is closed under arbitrary meets, but have proved a weaker statement that is is closed under finite meets: