15. Fréchet filter

The consideration below is about filters on a set U, but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorpic to algebras of sets on some set U.

Definition 71 $\{U \setminus X \mid X \text{ is a finite subset of } U\}$ *is called either* ***Fréchet filter*** *or* ***cofinite filter****.*

It is trivial that Fréchet filter is a filter.

Definition 72 *I will call* ***Fréchet f. o.*** *and denote* Ω *the filter object corresponding to the Fréchet filter.*

Proposition 35 $\operatorname{Cor}\Omega = \emptyset$.

Proof This can be deduced from the formula $\forall \alpha \in U \exists X \in \operatorname{up}\Omega : \alpha \notin X$. □

Theorem 69 $\max\{\mathcal{X} \in \mathfrak{F} \mid \operatorname{Cor}\mathcal{X} = \emptyset\} = \Omega$.

Proof Due the last proposition, enough to show that $\operatorname{Cor}\mathcal{X} = \emptyset \Rightarrow \mathcal{X} \subseteq \Omega$ for every f.o. $\mathcal{X}$.

Let $\operatorname{Cor}\mathcal{X} = \emptyset$ for some f.o. $\mathcal{X}$. Let $X \in \operatorname{up}\Omega$. We need to prove that $X \in \operatorname{up}\mathcal{X}$.

$X = U \setminus \{\alpha_0, \dots, \alpha_n\}$. $U \setminus \{\alpha_i\} \in \operatorname{up}\mathcal{X}$ because otherwise $\alpha_i \in \operatorname{Cor}\mathcal{X}$. So $X \in \operatorname{up}\mathcal{X}$. □

Theorem 70 $\Omega = \bigcup^{\mathfrak{F}}\{x \mid x \text{ is a non-trivial atomic f.o.}\}$.

Proof It follows from the facts that $\operatorname{Cor} x = \emptyset$ for every non-trivial atomic f.o. x, that $\mathfrak{F}$ is an atomistic lattice, and the previous theorem. □

Theorem 71 Cor *is the lower adjoint of* $\Omega \cup^{\mathfrak{F}} -$.

Proof Because both Cor and $\Omega \cup^{\mathfrak{F}} -$ are monotone, it is enough (theorem 8) to prove (for every filter objects $\mathcal{X}$ and $\mathcal{Y}$)

$$\mathcal{X} \subseteq \Omega \cup^{\mathfrak{F}} \operatorname{Cor}\mathcal{X} \quad \text{and} \quad \operatorname{Cor}(\Omega \cup^{\mathfrak{F}} \mathcal{Y}) \subseteq \mathcal{Y}.$$

$\operatorname{Cor}(\Omega \cup^{\mathfrak{F}} \mathcal{Y}) = \operatorname{Cor}\Omega \cup \operatorname{Cor}\mathcal{Y} = \emptyset \cup \operatorname{Cor}\mathcal{Y} = \operatorname{Cor}\mathcal{Y} \subseteq \mathcal{Y}$.

$\Omega \cup^{\mathfrak{F}} \operatorname{Cor}\mathcal{X} \supseteq \operatorname{Edg}\mathcal{X} \cup^{\mathfrak{F}} \operatorname{Cor}\mathcal{X} = \mathcal{X}$. □

Corollary 20 $\operatorname{Cor}\mathcal{X} = \mathcal{X} \setminus^* \Omega$ *for any f.o. on a set.*

Proof By the theorem 13. □

Corollary 21 $\operatorname{Cor}\bigcup^{\mathfrak{F}} S = \bigcup \langle\operatorname{Cor}\rangle S$ *for any set* S *of f.o. on a set.*

Proof By the theorem 12. □