Proof

⇒ Let a be an atom of $\mathfrak{A}$. $\operatorname{up} a \supseteq \partial a$ because $a \neq 0$. $\operatorname{up} a \subseteq \partial a$ because for any $K \in \mathfrak{A}$

$$K \in \operatorname{up} a \Leftrightarrow K \supseteq a \Rightarrow K \cap^{\mathfrak{A}} a \neq 0 \Leftrightarrow K \in \partial a.$$

⇐ Let $\operatorname{up} a = \partial a$. Then $a \neq 0$. Consequently for every $x \in \mathfrak{A}$ we have

$$\begin{aligned} 0 \subset x \subset a &\Rightarrow \\ x \cap^{\mathfrak{A}} a \neq 0 &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \partial a &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \operatorname{up} a &\Rightarrow \\ \operatorname{up} x \subseteq \operatorname{up} a &\Rightarrow \\ x \supseteq a. & \end{aligned}$$

So a is an atom of $\mathfrak{A}$.

□

10.1. Prime filtrator elements

Definition 63 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a down-aligned filtrator with least element 0. **Prime** filtrator elements are such $a \in \mathfrak{A}$ that $\operatorname{up} a$ is a free star.*

Proposition 30 *Let $(\mathfrak{A}; \mathfrak{Z})$ be a down-aligned filtrator with finitely join-closed core, where $\mathfrak{A}$ is a distributive lattice and $\mathfrak{Z}$ is a join-semilattice. Then atomic elements of this filtrator are prime.*

Proof Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X, Y \in \mathfrak{Z}$

$$\begin{aligned} X \cup^{\mathfrak{Z}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \cup^{\mathfrak{A}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \cup^{\mathfrak{A}} Y \supseteq a &\Leftrightarrow \\ (X \cup^{\mathfrak{A}} Y) \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ (X \cap^{\mathfrak{A}} a) \cup^{\mathfrak{A}} (Y \cap^{\mathfrak{A}} a) \neq 0 &\Leftrightarrow \\ X \cap^{\mathfrak{A}} a \neq 0 \vee Y \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ X \supseteq a \vee Y \supseteq a &\Leftrightarrow \\ X \in \operatorname{up} a \vee Y \in \operatorname{up} a. & \end{aligned}$$

□

The following theorem is essentially borrowed from [8]:

Theorem 52 *Let $\mathfrak{A}$ be a boolean lattice. Let a be a f.o. Then the following are equivalent:*