Proof

$$\begin{aligned}
\partial\mathcal{A} \subseteq \partial\mathcal{B} &\Leftrightarrow \\
\left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \operatorname{up}\mathcal{A} \right\} \subseteq \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus \operatorname{up}\mathcal{B} \right\} &\Leftrightarrow \\
\mathfrak{A} \setminus \operatorname{up}\mathcal{A} \subseteq \mathfrak{A} \setminus \operatorname{up}\mathcal{B} &\Leftrightarrow \\
\operatorname{up}\mathcal{A} \supseteq \operatorname{up}\mathcal{B} &\Leftrightarrow \\
\mathcal{A} \subseteq \mathcal{B}.&
\end{aligned}$$

□

Corollary 16 *∂ is a straight monotone map.*

Theorem 46 *If $\mathfrak{A}$ is a boolean lattice then $\partial \bigcup^{\mathfrak{F}} S = \bigcup \langle\partial\rangle S$.*

Proof For boolean lattices ∂ is an order embedding from the poset $\mathfrak{F}$ to the complete lattice $\mathcal{P}\mathfrak{A}$. So accordingly the lemma 2 it enough to prove that it exists $\mathcal{F} \in \mathfrak{F}$ such that $\partial\mathcal{F} = \bigcup \langle\partial\rangle S$. To prove this is enough to show that $0 \notin \bigcup \langle\partial\rangle S$ and

$$\forall A, B \in S : \left(A \cup^{\mathfrak{A}} B \in \bigcup \langle\partial\rangle S \Leftrightarrow A \in \bigcup \langle\partial\rangle S \vee B \in \bigcup \langle\partial\rangle S \right).$$

$0 \notin \bigcup \langle\partial\rangle S$ is obvious.

Let $A \cup^{\mathfrak{A}} B \in \bigcup \langle\partial\rangle S$. Then exists $Q \in \langle\partial\rangle S$ such that $A \cup^{\mathfrak{A}} B \in Q$. Then $A \in Q \vee B \in Q$, consequently $A \in \bigcup \langle\partial\rangle S \vee B \in \bigcup \langle\partial\rangle S$. Let now $A \in \bigcup \langle\partial\rangle S$. Then exists $Q \in \langle\partial\rangle S$ such that $A \in Q$, consequently $A \cup^{\mathfrak{A}} B \in Q$ and $A \cup^{\mathfrak{A}} B \in \bigcup \langle\partial\rangle S$. □

9.4. More about the lattice of filters

Theorem 47 *If $\mathfrak{A}$ is a distributive lattice with greatest element then $\mathfrak{F}$ is an atomic lattice.*

Proof Let $\mathcal{F} \in \mathfrak{F}$. Let choose (by Kuratowski's lemma) a maximal chain S from 0 to $\mathcal{F}$. Let $S' = S \setminus \{0\}$. $a = \bigcap^{\mathfrak{F}} S' \neq 0$ by properties of generalized filter bases (the corollary 12 which uses the fact that $\mathfrak{A}$ is a distributive lattice with least element). If $a \notin S$ then then the chain S can be extended adding there element a because $0 \subset a \subseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S'. Consequently (taking in account maximality of the chain) there are no $\mathcal{Y} \in \mathfrak{F}$ such that $0 \subset \mathcal{Y} \subset a$. So a is an atomic filter object. Obviously $a \subseteq \mathcal{F}$. □

Obvious 20 *If $\mathfrak{A}$ is a boolean lattice then $\mathfrak{F}$ is separable.*

Theorem 48 *If $\mathfrak{A}$ is a boolean lattice then $\mathfrak{F}$ is an atomistic lattice.*