Corollary 14 *If $\mathfrak{A}$ is a boolean lattice, $X \in \operatorname{up} \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A}$ for every $X \in \mathfrak{A}$, $\mathcal{A} \in \mathfrak{F}$.*

Corollary 15 *If $\mathfrak{A}$ is a boolean lattice, ∂ is an injection.*

Theorem 45 *If $\mathfrak{A}$ is a boolean lattice, then for any set $S \in \mathscr{P}\mathfrak{A}$ exists filter object $\mathcal{A}$ such that $\partial \mathcal{A} = S$ iff S is a free star.*

Proof

$\Rightarrow$ That $0 \notin S$ is obvious. For every $A, B \in \mathfrak{A}$

$$\begin{aligned} A \cup^{\mathfrak{A}} B \in S &\Leftrightarrow \\ (A \cup^{\mathfrak{A}} B) \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\ (A \cup^{\mathfrak{F}} B) \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\ (A \cap^{\mathfrak{F}} \mathcal{A}) \cup^{\mathfrak{F}} (B \cap^{\mathfrak{F}} \mathcal{A}) \neq 0 &\Leftrightarrow \\ A \cap^{\mathfrak{F}} \mathcal{A} \neq 0 \vee B \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\ A \in S \vee B \in S. & \end{aligned}$$

(taken into account the corollary 10 and theorem 23).

$\Leftarrow$ Let $0 \notin S$ and $\forall A, B \in S : (A \cup^{\mathfrak{A}} B \in S \Leftrightarrow A \in S \vee B \in S)$. Let $T = \left\{ \overline{X} \mid X \in \mathfrak{A} \setminus S \right\}$. We will prove that T is a filter.

$1 \in T$ because $0 \notin S$; so T is nonempty. To prove that T is a filter is enough to show that $\forall X, Y \in \mathfrak{A} : (X, Y \in T \Leftrightarrow X \cap^{\mathfrak{A}} Y \in T)$. In fact,

$$\begin{aligned} X, Y \in T &\Leftrightarrow \\ \overline{X}, \overline{Y} \notin S &\Leftrightarrow \\ \neg(\overline{X} \in S \vee \overline{Y} \in S) &\Leftrightarrow \\ \overline{X} \cup^{\mathfrak{A}} \overline{Y} \notin S &\Leftrightarrow \\ \overline{\overline{X} \cup^{\mathfrak{A}} \overline{Y}} \in T &\Leftrightarrow \\ X \cap^{\mathfrak{A}} Y \in T. & \end{aligned}$$

So T is a filter. Let $\operatorname{up} \mathcal{A} = T$ for some filter object $\mathcal{A}$.

To finish the proof we will show that $\partial \mathcal{A} = S$. In fact, for every $X \in \mathfrak{A}$

$$X \in \partial \mathcal{A} \Leftrightarrow \overline{X} \notin \operatorname{up} \mathcal{A} \Leftrightarrow \overline{X} \notin T \Leftrightarrow X \in S.$$

□

Proposition 29 *If $\mathfrak{A}$ is a boolean lattice then $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \partial \mathcal{A} \subseteq \partial \mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.*