Theorem 40 *Let $\mathfrak{F}$ be the set of filter objects over a boolean lattice $\mathfrak{A}$. $A \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \left\langle A \cap^{\mathfrak{F}} \right\rangle S$ for every $A \in \mathfrak{A}$ and every set $S \in \mathcal{P}\mathfrak{F}$.*

Proof Direct consequence of the lemma. □

8. Generalized filter base

Definition 58 ***Generalized filter base*** *is a filter base on the set $\mathfrak{F}$.*

Definition 59 *If S is a generalized filter base and $\mathcal{A} = \bigcap^{\mathfrak{F}} S$, then we will call S a generalized base of filter object $\mathcal{A}$.*

Theorem 41 *If $\mathfrak{A}$ is a distributive lattice and S is a generalized base of filter object $\mathcal{F}$ then for any element K of the base poset*

$$K \in \operatorname{up} \mathcal{F} \Leftrightarrow \exists \mathcal{L} \in S : \mathcal{L} \subseteq K.$$

Proof

$\Leftarrow$ Because $\mathcal{F} = \bigcap^{\mathfrak{F}} S$.

$\Rightarrow$ Let $K \in \operatorname{up} \mathcal{F}$. Then (taken in account distributivity of $\mathfrak{A}$ and that S is nonempty) exist $X_1, \ldots, X_n \in \bigcup \langle \operatorname{up} \rangle S$ such that $X_1 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} X_n = K$. Consequently (by theorem 29) $X_1 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} X_n = K$. Replacing every X_i with such $\mathcal{X}_i \in S$ that $X_i \in \operatorname{up} \mathcal{X}_i$ (this is obviously possible to do), we get a finite set $T_0 \subseteq S$ such that $\bigcap^{\mathfrak{F}} T_0 \subseteq K$. From this exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \bigcap^{\mathfrak{F}} T_0 \subseteq K$.

□

Corollary 12 *If $\mathfrak{A}$ is a distributive lattice with least element 0 and S is a generalized base of filter object $\mathcal{F}$ then $0 \in S \Leftrightarrow \mathcal{F} = 0$.*

Proof Substitute 0 as K. □

Theorem 42 *Let $\mathfrak{A}$ be a distributive lattice with least element 0 and S is a nonempty set of filter objects on $\mathfrak{A}$ such that $\mathcal{F}_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{F}_n \neq 0$ for every $\mathcal{F}_0, \ldots, \mathcal{F}_n \in S$. Then $\bigcap^{\mathfrak{F}} S \neq 0$.*

Proof Consider the set

$$S' = \left\{ \mathcal{F}_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} \mathcal{F}_n \mid \mathcal{F}_0, \ldots, \mathcal{F}_n \in S \right\}.$$

Obviously S' is nonempty and finitely meet-closed. So S' is a generalized filter base. Obviously $0 \notin S'$. So by properties of generalized filter bases $\bigcap^{\mathfrak{F}} S' \neq 0$. But obviously $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} S'$. So $\bigcap^{\mathfrak{F}} S \neq 0$. □