**Proof** Taking in account the lemma it is enough to prove that exists $\mathcal{F} \in \mathfrak{F}$ such that $\operatorname{up} \mathcal{F} = \bigcap^{\mathcal{P}\mathfrak{A}} \langle \operatorname{up} \rangle S$, that is that $R = \bigcap^{\mathcal{P}\mathfrak{A}} \langle \operatorname{up} \rangle S$ is a filter.

$R$ is nonempty because $1 \in R$. Let $A, B \in R$; then $\forall \mathcal{F} \in S : A, B \in \operatorname{up} \mathcal{F}$, consequently $\forall \mathcal{F} \in S : A \cap^{\mathfrak{A}} B \in \operatorname{up} \mathcal{F}$. Consequently $A \cap^{\mathfrak{A}} B \in \bigcap^{\mathcal{P}\mathfrak{A}} \langle \operatorname{up} \rangle S = R$. So $R$ is a filter base. Let $X \in R$ and $X \subseteq Y \in \mathfrak{A}$; then $\forall \mathcal{F} \in S : X \in \operatorname{up} \mathcal{F}$; $\forall \mathcal{F} \in S : Y \in \operatorname{up} \mathcal{F}$; $Y \in R$. So $R$ is an upper set. □

**Corollary 8** *If $\mathfrak{A}$ is a meet-semilattice with greatest element $1$ then $\mathfrak{F}$ is a complete lattice.*

**Corollary 9** *If $\mathfrak{A}$ is a meet-semilattice with greatest element $1$ then for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$*

$$\operatorname{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}.$$

**Theorem 33** *If $\mathfrak{A}$ is a join-semilattice then $\mathfrak{F}$ is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$*

$$\operatorname{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}.$$

**Proof** Taking in account the lemma it is enough to prove that $R = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}$ is a filter.

$R$ is nonempty because exist $X \in \operatorname{up} \mathcal{A}$ and $Y \in \operatorname{up} \mathcal{B}$ and $R \ni X \cup^{\mathfrak{A}} Y$.

Let $A, B \in R$. Then $A, B \in \operatorname{up} \mathcal{A}$; so exists $C \in \operatorname{up} \mathcal{A}$ such that $C \subseteq A \wedge C \subseteq B$. Analogously exists $D \in \operatorname{up} \mathcal{B}$ such that $D \subseteq A \wedge D \subseteq B$. Let $E = C \cup^{\mathfrak{A}} D$. Then $E \in \operatorname{up} \mathcal{A}$ and $E \in \operatorname{up} \mathcal{B}$; $E \in R$ and $E \subseteq A \wedge E \subseteq B$. So $R$ is a filter base.

That $R$ is an upper set is obvious. □

**Theorem 34** *If $\mathfrak{A}$ is a distributive lattice then for $S \in \mathcal{P}\mathfrak{F} \setminus \{\emptyset\}$*

$$\operatorname{up} \bigcap{}^{\mathfrak{F}} S = \left\{ K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \;\middle|\; K_i \in \bigcup \langle \operatorname{up} \rangle S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N} \right\}.$$

**Proof** Let's denote the right part of the equality to be proven as $R$. First we will prove that $R$ is a filter. $R$ is nonempty because $S$ is nonempty.

Let $A, B \in R$. Then $A = X_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} X_k$, $B = Y_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} Y_l$ where $X_i, Y_j \in \bigcup \langle \operatorname{up} \rangle S$. So

$$A \cap^{\mathfrak{A}} B = X_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} X_k \cap^{\mathfrak{A}} Y_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} Y_l \in R.$$

Let $R \ni C \supseteq A$. Consequently (distributivity used)

$$C = C \cup^{\mathfrak{A}} A = (C \cup^{\mathfrak{A}} X_0) \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} (C \cup^{\mathfrak{A}} X_k).$$

$X_i \in \operatorname{up} P$ for some $P \in S$; $C \cup^{\mathfrak{A}} X_i \in \operatorname{up} P$; consequently $C \in \operatorname{up} P$; $C \in R$.

We have proved that $R$ is a filter base and an upper set. So $R$ is a filter.

Consequently the statement of our theorem is equivalent to $\bigcap^{\mathfrak{F}} S = \operatorname{up}^{-1} R$.

Let $\mathcal{A} \in S$. Then $\operatorname{up} \mathcal{A} \in \langle \operatorname{up} \rangle S$; $\operatorname{up} \mathcal{A} \subseteq \bigcup \langle \operatorname{up} \rangle S$;

$$R \supseteq \left\{ K_0 \cap^{\mathfrak{A}} \ldots \cap^{\mathfrak{A}} K_n \;\middle|\; K_i \in \operatorname{up} \mathcal{A} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N} \right\} = \operatorname{up} \mathcal{A}.$$