(3)⇒(2) Let
$$\forall X,Y \in \mathfrak{A} : (X,Y \in F \Leftrightarrow X \cap Y \in F).$$

Then $\forall X,Y \in F : X \cap Y \in F$. Let $X \in F$ and $X \subseteq Y \in \mathfrak{A}$. Then $X \cap Y = X \in F$. Consequently $X,Y \in F$. So F is an upper set.

□

Proposition 20 *Let $\mathfrak{A}$ be a meet-semilattice. Let S be a filter base. If $A_0,\dots,A_n \in S$ $(n \in \mathbb{N})$, then*
$$\exists C \in S : C \subseteq A_0 \cap \ldots \cap A_n.$$

Proof It can be easily proved by induction. □

Proposition 21 *If $\mathfrak{A}$ is a meet-semilattice and S is a filter base, $A \in \mathfrak{A}$, then $\langle A\cap\rangle\, S$ is also a filter base.*

Proof $\langle A\cap\rangle\, S \neq \emptyset$ because $S \neq \emptyset$.

Let $X,Y \in \langle A\cap\rangle\, S$. Then $X = A \cap X'$ and $Y = A \cap Y'$ where $X',Y' \in S$. Exists $Z' \in S$ such that $Z' \subseteq X' \cap Y'$. So $X \cap Y = A \cap X' \cap Y' \supseteq A \cap Z' \in \langle A\cap\rangle\, S$.
□

5.3. Characterization of finitely meet-closed filtrators

Theorem 29 *The following are equivalent for a filtrator $(\mathfrak{A};\mathfrak{Z})$ whose core is a meet-semilattice such that $\forall a \in \mathfrak{A} : \operatorname{up} a \neq \emptyset$:*

1. *The filtrator is finitely meet-closed.*

2. *$\operatorname{up} a$ is a filter on $\mathfrak{Z}$ for every $a \in \mathfrak{A}$.*

Proof

(1)⇒(2) Let $X,Y \in \operatorname{up} a$. Then $X \cap^{\mathfrak{Z}} Y = X \cap^{\mathfrak{A}} Y \supseteq a$. That $\operatorname{up} a$ is an upper set is obvious. So taking in account that $\operatorname{up} a \neq \emptyset$, $\operatorname{up} a$ is a filter.

(2)⇒(1) It is enough to prove that $a \subseteq A,B \Rightarrow a \subseteq A \cap^{\mathfrak{Z}} B$ for every $A,B \in \mathfrak{A}$. Really:
$$a \subseteq A,B \Rightarrow A,B \in \operatorname{up} a \Rightarrow A \cap^{\mathfrak{Z}} B \in \operatorname{up} a \Rightarrow a \subseteq A \cap^{\mathfrak{Z}} B.$$

□

6. Filter objects

I want to equate principal filters (see below) with the elements of the base poset. Such thing can be done using the principles described in the appendix Appendix B. The formal definitions follow.