Obvious 14 *A nonempty chain is a filter base.*

Definition 52 ***Upper set*** *is a subset F of $\mathfrak{A}$ such that*

$$\forall X \in F, Y \in \mathfrak{A} : (Y \supseteq X \Rightarrow Y \in F).$$

Definition 53 *Filter is a subset of $\mathfrak{A}$ which is both filter base and upper set. I will denote the set of filters $\mathfrak{f}$.*

Proposition 18 *If 1 is the maximal element of $\mathfrak{A}$ then $1 \in F$ for any filter F.*

Proof If $1 \notin F$ then $\forall K \in \mathfrak{A} : K \notin F$ and so F is empty what is impossible. □

Proposition 19 *Let S be a filter base. If $A_0, \ldots, A_n \in S$ $(n \in \mathbb{N})$, then*

$$\exists C \in S : (C \subseteq A_0 \wedge \ldots \wedge C \subseteq A_n).$$

Proof It can be easily proved by induction. □

The dual of filters is called **ideals**. We do not use ideals in this work however.

5.2. Filters on meet-semilattice

Theorem 28 *If $\mathfrak{A}$ is a meet-semilattice and F is a nonempty subset of $\mathfrak{A}$ then the following conditions are equivalent:*

1. *F is a filter.*

2. *$\forall X, Y \in F : X \cap Y \in F$ and F is an upper set.*

3. *$\forall X, Y \in \mathfrak{A} : (X, Y \in F \Leftrightarrow X \cap Y \in F)$.*

Proof

(1)⇒(2) Let F be a filter. Then F is an upper set. If $X, Y \in F$ then $Z \subseteq X \wedge Z \subseteq Y$ for some $Z \in F$. Because F is an upper set and $Z \subseteq X \cap Y$ then $X \cap Y \in F$.

(2)⇒(1) Let $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set. We need to prove that F is a filter base. But it is obvious taking $Z = X \cap Y$ (we have also taken in account that $F \neq \emptyset$).

(2)⇒(3) Let $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set. Then

$$\forall X, Y \in \mathfrak{A} : (X, Y \in F \Rightarrow X \cap Y \in F).$$

Let $X \cap Y \in F$; then $X, Y \in F$ because F is an upper set.