$\Rightarrow$ Let $a \neq b$ for example $a \nsubseteq b$. Then $a \cap b \subset a$; $\operatorname{atoms} a \supset \operatorname{atoms}(a \cap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ and thus $\operatorname{atoms} a \neq \operatorname{atoms} b$.

Let $\operatorname{atoms} a \neq \operatorname{atoms} b$ for example $\operatorname{atoms} a \nsubseteq \operatorname{atoms} b$. Then $\operatorname{atoms}(a \cap b) = \operatorname{atoms} a \cap \operatorname{atoms} b \subset \operatorname{atoms} a$ and thus $a \cap b \subset a$ and so $a \nsubseteq b$ consequently $a \neq b$.

□

Proposition 14 *Any atomistic poset is atomically separable.*

Proof We need to prove that $\operatorname{atoms} a = \operatorname{atoms} b \Rightarrow a = b$. But it is obvious because

$$a = \bigcup \operatorname{atoms} a \quad \text{and} \quad b = \bigcup \operatorname{atoms} b.$$

□

Theorem 20 *If a lattice with least element is atomic and separable then it is atomistic.*

Proof Suppose the contrary that is $a \supset \bigcup \operatorname{atoms} a$. Then, because our lattice is separable, exists $c \in \mathfrak{A}$ such that $c \cap a \neq 0$ and $c \cap \bigcup \operatorname{atoms} a = 0$. There exist atom $d \subseteq c$ such that $d \subseteq c \cap a$. $d \cap \bigcup \operatorname{atoms} a \subseteq c \cap \bigcup \operatorname{atoms} a = 0$. But $d \in \operatorname{atoms} a$. Contradiction. □

Theorem 21 *Any atomistic lattice is atomically separable.*

Proof Let $\mathfrak{A}$ be an atomistic lattice. Let $a, b \in \mathfrak{A}$, $a \subset b$. Then $\bigcup \operatorname{atoms} a \subset \bigcup \operatorname{atoms} b$ and consequently $\operatorname{atoms} a \subset \operatorname{atoms} b$. □

Theorem 22 *Let $\mathfrak{A}$ be an atomic meet-semilattice with least element. Then the following statements are equivalent:*

1. *$\mathfrak{A}$ is separable.*
2. *$\mathfrak{A}$ is atomically separable.*
3. *$\mathfrak{A}$ conforms to Wallman's disjunction property.*
4. *$\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{0\} : (c \asymp a \wedge c \subseteq b))$.*

Proof

(1)⇔(3)⇔(4) Proved above.

(2)⇒(4) Let our semilattice be atomically separable. Let $a \subset b$. Then $\operatorname{atoms} a \subset \operatorname{atoms} b$ and so exists $c \in \operatorname{atoms} b$ such that $c \notin \operatorname{atoms} a$. $c \neq 0$ and $c \subseteq b$; $c \nsubseteq a$, from which (taking in account that c is an atom) $c \subseteq b$ and $c \cap a = 0$. So our semilattice conforms to the formula (4).