1. Let $g(b) = \max\left\{ x \in \mathfrak{A} \mid fx \subseteq b \right\}$ for every $b \in \mathfrak{B}$. Then

$$x \subseteq gy \Leftrightarrow x \subseteq \max\left\{ x \in \mathfrak{A} \mid fx \subseteq y \right\} \Rightarrow fx \subseteq y$$

(because f is monotone) and

$$x \subseteq gy \Leftrightarrow x \subseteq \max\left\{ x \in \mathfrak{A} \mid fx \subseteq y \right\} \Leftarrow fx \subseteq y.$$

So $fx \subseteq y \Leftrightarrow x \subseteq gy$ that is f is the lower adjoint of g.

2. We have

$$g(b) = \max\left\{ x \in \mathfrak{A} \mid fx \subseteq b \right\} \quad \Leftrightarrow$$
$$fgb \subseteq b \wedge \forall x \in \mathfrak{A} : (fx \subseteq b \Rightarrow x \subseteq gb)$$

what is true by properties of adjoints.

□

Theorem 12 *Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.*

1. If f is an upper adjoint, f preserves all existing infima in $\mathfrak{A}$.

2. If $\mathfrak{A}$ is a complete lattice and f preserves all infima, then f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.

3. If f is a lower adjoint, f preserves all existing suprema in $\mathfrak{A}$.

4. If $\mathfrak{A}$ is a complete lattice and f preserves all suprema, then f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.

Proof We will prove only first two items because the rest items are similar.

1. Let $S \in \mathcal{P}\mathfrak{A}$ and $\bigcap S$ exists. $f \bigcap S$ is a lower bound for $\langle f \rangle S$ because f is order-preserving. If a is a lower bound for $\langle f \rangle S$ then $\forall x \in S : a \subseteq fx$ that is $\forall x \in S : x \subseteq ga$ where g is the lower adjoint of f. Thus $\bigcap S \subseteq ga$ and hence $f \bigcap S \subseteq a$. So $f \bigcap S$ is the greatest lower bound for $\langle f \rangle S$.

2. Let $\mathfrak{A}$ be a complete lattice and f preserves all infima. Let $g(a) = \bigcap\left\{ x \in \mathfrak{A} \mid fx \supseteq a \right\}$. Since f preserves infima, we have

$$f(g(a)) = \bigcap\left\{ f(x) \mid x \in \mathfrak{A}, f(x) \supseteq a \right\} \supseteq a.$$

$g(f(b)) = \bigcap\left\{ x \in \mathfrak{A} \mid fx \supseteq fb \right\} \subseteq b$.

Obviously f is monotone and thus g is also monotone.

So f is the upper adjoint of g.

□

Corollary 3 *Let f be a function from a complete lattice $\mathfrak{A}$ to a poset $\mathfrak{B}$. Then:*

1. f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all infima in $\mathfrak{A}$.

2. f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all suprema in $\mathfrak{A}$.