Theorem 9

1. $f^* \circ f_* \circ f^* = f^*$.

2. $f_* \circ f^* \circ f_* = f_*$.

Proof

1. Let $x \in \mathfrak{A}$. We have $x \subseteq^{\mathfrak{A}} f_* f^* x$; consequently $f^* x \subseteq^{\mathfrak{B}} f^* f_* f^* x$. On the other hand, $f^* f_* f^* x \subseteq^{\mathfrak{B}} f^* x$. So $f^* f_* f^* x = f^* x$.

2. Analogously.

□

Proposition 6 *$f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.*

Proof $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

Theorem 10 *Each of two adjoints is uniquely determined by the other.*

Proof Let p and q be both upper adjoints of f. We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \subseteq p(y) \Leftrightarrow f(x) \subseteq y \Leftrightarrow x \subseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \subseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \subseteq p(y)$. So $p(y) = q(y)$. □

Theorem 11 *Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.*

1. Both:

   1. If f is monotone and $g(b) = \max \{ x \in \mathfrak{A} \mid fx \subseteq b \}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f.

   2. If $g \colon \mathfrak{B} \to \mathfrak{A}$ is the upper adjoint of f then $g(b) = \max \{ x \in \mathfrak{A} \mid fx \subseteq b \}$ for every $b \in \mathfrak{B}$.

2. Both:

   1. If f is monotone and $g(b) = \min \{ x \in \mathfrak{A} \mid fx \supseteq b \}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f.

   2. If $g \colon \mathfrak{B} \to \mathfrak{A}$ is the lower adjoint of f then $g(b) = \min \{ x \in \mathfrak{A} \mid fx \supseteq b \}$ for every $b \in \mathfrak{B}$.

Proof We will prove only the first as the second is its dual.