Proof See [6]. □

Remark 4 See [7] for a more strong result.

Theorem 6 *The center of a bounded distributive lattice constitutes its sublattice.*

Proof Let $\mathfrak{A}$ be a bounded distributive lattice and $Z(\mathfrak{A})$ is its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \cup \bar{b}$ is the complement of $a \cap b$ because

$$(a \cap b) \cap (\bar{a} \cup \bar{b}) = (a \cap b \cap \bar{a}) \cup (a \cap b \cap \bar{b}) = 0 \cup 0 = 0 \quad \text{and}$$
$$(a \cap b) \cup (\bar{a} \cup \bar{b}) = (a \cup \bar{a} \cup \bar{b}) \cap (b \cup \bar{a} \cup \bar{b}) = 1 \cap 1 = 1.$$

So $a \cap b$ is complemented, analogously $a \cup b$ is complemented. □

Theorem 7 *The center of a bounded distributive lattice constitutes a boolean lattice.*

Proof Because it is a distributive complemented lattice. □

2.5. Galois connections

See [1] and [5] for more detailed treatment of Galois connections.

Definition 17 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two posets. A* ***Galois connection*** *between $\mathfrak{A}$ and $\mathfrak{B}$ is a pair of functions $f = (f^*; f_*)$ with $f^* : \mathfrak{A} \to \mathfrak{B}$ and $f_* : \mathfrak{B} \to \mathfrak{A}$ such that:*

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (f^* x \subseteq^{\mathfrak{B}} y \Leftrightarrow x \subseteq^{\mathfrak{A}} f_* y).$$

f_* is called ***upper adjoint*** *of f^* and f^* is called* ***lower adjoint*** *of f_*.*

Theorem 8 *A pair $(f^*; f_*)$ of functions $f^* : \mathfrak{A} \to \mathfrak{B}$ and $f_* : \mathfrak{B} \to \mathfrak{A}$ is a Galois connection iff both of the following:*

1. *f^* and f_* are monotone.*

2. *$x \subseteq^{\mathfrak{A}} f_* f^* x$ and $f^* f_* y \subseteq^{\mathfrak{B}} y$ for every $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$.*

Proof

$\Rightarrow$ 2. $x \subseteq^{\mathfrak{A}} f_* f^* x$ since $f^* x \subseteq^{\mathfrak{B}} f^* x$; $f^* f_* y \subseteq^{\mathfrak{B}} y$ since $f_* y \subseteq^{\mathfrak{A}} f_* y$.

1. Let $a, b \in \mathfrak{A}$ and $a \subseteq^{\mathfrak{A}} b$. Then $a \subseteq^{\mathfrak{A}} b \subseteq^{\mathfrak{A}} f_* f^* b$. So by definition $f^* a \subseteq f^* b$ that is f^* is monotone. Analogously f_* is monotone.

$\Leftarrow$ $f^* x \subseteq^{\mathfrak{B}} y \Rightarrow f_* f^* x \subseteq^{\mathfrak{A}} f_* y \Rightarrow x \subseteq^{\mathfrak{A}} f_* y$. The other direction is analogous.

□