2.3. Difference and complement

Definition 8 *Let $\mathfrak{A}$ be a distributive lattice with least element 0. The* ***difference*** *(denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \cap c = 0$ and $a \cup b = b \cup c$. I will call b* ***substractive*** *from a when $a \setminus b$ exists.*

Theorem 4 *If $\mathfrak{A}$ is a distributive lattice with least element 0, there exists no more than one difference of elements $a, b \in \mathfrak{A}$.*

Proof Let c and d are both differences $a \setminus b$. Then $b \cap c = b \cap d = 0$ and $a \cup b = b \cup c = b \cup d$. So

$$c = c \cap (b \cup c) = c \cap (b \cup d) = (c \cap b) \cup (c \cap d) = 0 \cup (c \cap d) = c \cap d.$$

Analogously, $d = d \cap c$. Consequently $c = c \cap d = d \cap c = d$. □

Definition 9 *I will call b* ***complementive*** *to a when there exists $c \in \mathfrak{A}$ such that $b \cap c = 0$ and $b \cup c = a$.*

Proposition 2 *b is complementive to a iff b is substractive from a and $b \subseteq a$.*

Proof

$\Leftarrow$ Obvious.

$\Rightarrow$ We deduce $b \subseteq a$ from $b \cup c = a$. Thus $a \cup b = a = b \cup c$.

□

Proposition 3 *If b is complementive to a then $(a \setminus b) \cup b = a$.*

Proof Because $b \subseteq a$ by the previous proposition. □

Definition 10 *Let $\mathfrak{A}$ be a bounded distributive lattice. The* ***complement*** *(denoted $\bar{a}$) of element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \cap b = 0$ and $a \cup b = 1$.*

Proposition 4 *If $\mathfrak{A}$ is a bounded distributive lattice then $\bar{a} = 1 \setminus a$.*

Proof $b = \bar{a} \Leftrightarrow b \cap a = 0 \wedge b \cup a = 1 \Leftrightarrow b \cap a = 0 \wedge 1 \cup a = a \cup b \Leftrightarrow b = 1 \setminus a$. □

Corollary 2 *If $\mathfrak{A}$ is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.*

Definition 11 *An element of bounded distributive lattice is called* ***complemented*** *when its complement exists.*

Definition 12 *A distributive lattice is a* ***complemented lattice*** *iff every its element is complemented.*