Proof. f is a discrete monovalued reloid. Thus $f = F|_{\mathrm{dom}\, f}$ where F is a discrete monovalued reloids. Thus f is discrete. □

Example 84. There exist two atomic reloids whose composition is non-atomic and non-empty.

Proof. Let a is a non-trivial atomic filter object on $\mathbb{N}$ and $x \in \mathbb{N}$. Then

$(a \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}}\{x\}) \circ (\uparrow^{\mathbb{N}}\{x\} \times^{\mathsf{RLD}} a) = \bigcap \{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}((A \times \{x\}) \circ (\{x\} \times A)) \mid A \in \mathrm{up}\, a\} = \bigcap \{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(A \times A) \mid A \in \mathrm{up}\, a\} = a \times^{\mathsf{RLD}} a$

is non-atomic despite of $a \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}}\{x\}$ and $\uparrow^{\mathbb{N}}\{x\} \times^{\mathsf{RLD}} a$ are atomic. □

Example 85. There exists non-monovalued atomic reloid.

Proof. From the previous example follows that the atomic reloid $\uparrow^{\mathbb{N}}\{x\} \times^{\mathsf{RLD}} a$ is not monovalued. □

Example 86. $\mathcal{A} \geqslant_2 \mathcal{B} \wedge \mathcal{B} \geqslant_2 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. (proof idea by Andreas Blass, rewritten using reloids by me)

Let u_n, h_n with n ranging over the set $\mathbb{Z}$ are sequences of atomic f.o. on $\mathbb{N}$ and functions $\mathbb{N} \to \mathbb{N}$ such that $\langle \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})} h_n \rangle u_{n+1} = u_n$ and u_n are pairwise non-isomorphic. (See [1] for a proof that such ultrafilters and functions exist.)

$\mathcal{A} \overset{\mathrm{def}}{=} \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\}$; $\mathcal{B} \overset{\mathrm{def}}{=} \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\}$.

Let the **Set**-morphisms $f, g: \mathbb{Z} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ are defined by the formulas $f(n; x) = (n; h_{2n} x)$ and $g(n; x) = (n - 1; h_{2n-1} x)$.

Using the fact that every function induces a complete funcoid and a lemma above we get:

$\langle f \rangle \mathcal{A} = \bigcup \langle \langle \uparrow f \rangle \rangle \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\} = \mathcal{B}$.

$\langle g \rangle \mathcal{B} = \bigcup \langle \langle \uparrow g \rangle \rangle \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathbb{Z}}\{n - 1\} \times^{\mathsf{RLD}} u_{2n-1} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\} = \mathcal{A}$.

It remains to show that $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic.

Let $X \in \mathrm{up}(\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1})$ for some $n \in \mathbb{Z}$. Then if $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \cap \mathcal{A}$ is an atomic f.o. we have $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \cap \mathcal{A} = \uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1}$ and thus by the theorem 78 is isomorphic to u_{2n+1}.

If $X \notin \mathrm{up}(\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1})$ for every $n \in \mathbb{Z}$ then $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \mathrm{up}(\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1})$ and thus $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \mathrm{up}\, \mathcal{A}$ and thus $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \cap \mathcal{A} = \emptyset$.

We have also $(\uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} \mathbb{N}) \cap \mathcal{B} = (\uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} \mathbb{N}) \cap \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\} = \bigcup \{(\uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} \mathbb{N}) \cap (\{n\} \times^{\mathsf{RLD}} u_{2n}) \mid n \in \mathbb{Z}\} = \uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} u_0$ (an atomic f.o.).

Thus every atomic f.o. generated as intersecting $\mathcal{A}$ with a principal f.o. $\uparrow^{\mathbb{Z} \times \mathbb{N}} X$ is isomorphic to some u_{2n+1} and thus is not isomorphic to u_0. By the lemma it follows that $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic. □

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