Proof. Suppose that f and g are two different bijective reloids from $\mathcal{A}$ to $\mathcal{B}$. Then $g^{-1}\circ f$ is not the identity reloid (otherwise $g^{-1}\circ f = I^{\mathsf{RLD}}_{\operatorname{dom} f}$ and so $f = g$). But $g^{-1}\circ f$ is a bijective reloid (as a composition of bijective reloids) from $\mathcal{A}$ to $\mathcal{A}$ what is impossible. □

4 Rudin-Keisler equivalence and Rudin-Keisler order

Theorem 61. Atomic filter objects a and b (with possibly different bases) are isomorphic iff $a \geqslant b \wedge b \geqslant a$.

Proof. Let $a \geqslant b \wedge b \geqslant a$. Then there are a monovalued reloids f and g such that $\operatorname{dom} f = a$ and $\operatorname{im} f = b$ and $\operatorname{dom} g = b$ and $\operatorname{im} g = a$. Thus $g\circ f$ is a monovalued morphism from a to a. By the above we have $g\circ f = I^{\mathsf{RLD}}_a$ so $g = f^{-1}$ and $f^{-1}\circ f = I^{\mathsf{RLD}}_a$ so f is monovalued. Thus f is an injective monovalued reloid from a to b and thus a and b are isomorphic. □

The last theorem cannot be generalized from atomic f.o. to arbitrary f.o., as it's shown by the following two examples:

Example 62. $\mathcal{A} \geqslant_1 \mathcal{B} \wedge \mathcal{B} \geqslant_1 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. Consider $\mathcal{A} = \uparrow^{\mathbb{R}}[0;1]$ and $\mathcal{B} = \bigcap \{\uparrow^{\mathbb{R}}[0;1+\varepsilon) \mid \varepsilon > 0\}$. Then the function $f = \{(x; x/2) \mid x \in \mathbb{R}\}$ witnesses both inequalities $\mathcal{A} \geqslant_1 \mathcal{B}$ and $\mathcal{B} \geqslant_1 \mathcal{A}$. But these filters cannot be isomorphic because only one of them is principal. □

Lemma 63. Let f_0 and f_1 are **Set**-morphisms. Let $f(x;y) = (f_0 x; f_1 y)$ for a function f. Then $\langle \uparrow^{\mathsf{FCD}(\operatorname{Dst} f_0; \operatorname{Dst} f_1)} f\rangle (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \langle \uparrow f_0\rangle \mathcal{A} \times^{\mathsf{RLD}} \langle \uparrow f_1\rangle \mathcal{B}$.

Proof. $\langle \uparrow^{\mathsf{FCD}(\operatorname{Dst} f_0; \operatorname{Dst} f_1)} f\rangle (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \langle \uparrow^{\mathsf{FCD}(\operatorname{Dst} f_0; \operatorname{Dst} f_1)} f\rangle \bigcap \{\uparrow^{\operatorname{Src} f_0 \times \operatorname{Src} f_1} (A \times B) \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} = \bigcap \{\uparrow^{\operatorname{Src} f_0 \times \operatorname{Src} f_1} \langle f\rangle (A \times B) \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} = \bigcap \{\uparrow^{\operatorname{Dst} f_0 \times \operatorname{Dst} f_1} (\langle f_0\rangle A \times \langle f_1\rangle B) \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} = \bigcap \{\uparrow^{\operatorname{Dst} f_0} \langle f_0\rangle A \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f_1} \langle f_1\rangle B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} =$ (theorem 164?? in [3])$= \bigcap \{\uparrow^{\operatorname{Dst} f_0} \langle f_0\rangle A \mid A \in \operatorname{up} \mathcal{A}\} \times^{\mathsf{RLD}} \bigcap \{\uparrow^{\operatorname{Dst} f_1} \langle f_1\rangle B \mid A \in \operatorname{up} \mathcal{B}\} = \langle \uparrow f_0\rangle \mathcal{A} \times^{\mathsf{RLD}} \langle \uparrow f_1\rangle \mathcal{B}$. □

Lemma 64. If an f.o. $\mathcal{A}$ is isomorphic to an f.o. $\mathcal{B}$ then if X is a set and $\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}$ is an atomic f.o., then there exists a set Y such that $\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}$ is an atomic f.o. isomorphic to $\uparrow^{\operatorname{Base}(\mathcal{A})} Y \cap \mathcal{B}$. [FIXME: See the book for a corrected proof.]

Proof. Let $\mathcal{A}$ is isomorphic to $\mathcal{B}$. Then there are sets $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$. So there are a bijection $f: \mathscr{P} A \cap \operatorname{up} \mathcal{A} \to \mathscr{P} B \cap \operatorname{up} \mathcal{B}$ such that $\mathcal{B} = \langle f\rangle \mathcal{A}$.

[FIXME: ?? equality is wrong.]

$\operatorname{up}(\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}) = \operatorname{up}(\uparrow^{\operatorname{Base}(\mathcal{A})} (X \cap A) \cap \mathcal{A}) = ?? = \langle X \cap A \cap\rangle \operatorname{up} \mathcal{A} = \langle X \cap\rangle (\mathscr{P} A \cap \operatorname{up} \mathcal{A})$.

Thus $\langle\langle f\rangle\rangle \operatorname{up}(\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}) = \langle\langle f\rangle\rangle \langle X \cap\rangle (\mathscr{P} A \cap \operatorname{up} \mathcal{A}) = \langle f(X) \cap\rangle \langle\langle f\rangle\rangle (\mathscr{P} A \cap \operatorname{up} \mathcal{A}) = \langle f(X) \cap\rangle (\mathscr{P} B \cap \operatorname{up} \mathcal{B}) = \langle f(X) \cap B \cap\rangle \operatorname{up} \mathcal{B} = \langle f(X) \cap\rangle \operatorname{up} \mathcal{B} = \operatorname{up}(\uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B})$.

So $\langle f\rangle (\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}) = \bigcap \langle \uparrow^{\operatorname{Base}(\mathcal{B})}\rangle \langle\langle f\rangle\rangle \operatorname{up}(\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}) = \bigcap \langle \uparrow^{\operatorname{Base}(\mathcal{B})}\rangle \operatorname{up}(\uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B}) = \uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B}$.

Finally we have $\uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B}$ is isomorphic to $\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}$ from the last equality. □

Theorem 65. Let f is a monovalued injective reloid. Then f is isomorphic to the f.o. $\operatorname{dom} f$.

Proof. Let f is a monovalued injective reloid. There exists a bijection $F \in \operatorname{up} f$. Consider the bijective function $p = \{(x; Fx) \mid x \in \operatorname{dom} F\}$.

$\langle p\rangle \operatorname{dom} F = F$ and consequently $\langle p\rangle \operatorname{dom} f = \bigcap \{\uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} \langle p\rangle \operatorname{dom} K \mid K \in \operatorname{up} f\} = \bigcap \{\uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} \langle p\rangle \operatorname{dom}(K \cap F) \mid K \in \operatorname{up} f\} = \bigcap \{\uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} (K \cap F) \mid K \in \operatorname{up} f\} = \bigcap \{\uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} K \mid K \in \operatorname{up} f\} = f$. Thus p witnesses that f is isomorphic to the f.o. $\operatorname{dom} f$. □

Corollary 66. A monovalued injective reloid with atomic domain is atomic.