Lemma 54. $A \notin \mu$.

Proof. Suppose $A \in \mu$.

Since $A \in \mu$ we have $B_0 \in \mu$ or $B_1 \in \mu$.

So either $B_0 \cap \langle f \rangle B_0 \subseteq B_2$ or $B_1 \cap \langle f \rangle B_1 \subseteq B_2$. As such by the lemma 50 we have $B_2 \in \mu$. This is incompatible with $B_2 \cap \langle f \rangle B_2 = \emptyset$. So we got a contradiction. □

Let C be the set of points x which are not periodic but $f^n(x)$ is periodic for some positive n.

Lemma 55. $C \notin \mu$.

Proof. Let β be a function $C \rightarrow \mathbb{N}$ such that $\beta(x)$ is the least $n \in \mathbb{N}$ such that $f^n(x)$ is periodic.

Let $C_0 = \{x \in C \mid \beta(x) \text{ is even}\}$ and $C_1 = \{x \in C \mid \beta(x) \text{ is odd}\}$.

Obviously $C_j \cap \langle f \rangle C_j = \emptyset$ for $j = 0, 1$. Hence by the lemma 50 we have $C_0, C_1 \notin \mu$ and thus $C = C_0 \cup C_1 \notin \mu$. □

Let E be the set of $x \in I$ such that for no $n \in \mathbb{N}$ we have $f^n(x)$ periodic.

Lemma 56. Let $x, y \in E$ are such that $f^i(x) = f^j(y)$ and $f^{i'}(x) = f^{j'}(y)$ for some $i, j, i', j' \in \mathbb{N}$. Then $i - j = i' - j'$.

Proof. $i \mapsto f^i(x)$ is a bijection.

So $y = f^{i-j}(y)$ and $y = f^{i'-j'}(y)$. Thus $f^{i-j}(y) = f^{i'-j'}(y)$ and so $i - j = i' - j'$. □

Lemma 57. $E \notin \mu$.

Proof. Let $D' \subseteq E$ be a subset of E with exactly one element from each equivalence class of the relation $\sim$ on E.

Define the function $\gamma: E \rightarrow \mathbb{Z}$ as follows. Let $x \in E$. Let y be the unique element of D' such that $x \sim y$. Choose $i, j \in \mathbb{N}$ such that $f^i(y) = f^j(x)$. Let $\gamma(x) = i - j$. By the last lemma, γ is well-defined.

It is clear that if $x \in E$ then $f(x) \in E$ and moreover $\gamma(f(x)) = \gamma(x) + 1$.

Let $E_0 = \{x \in E \mid \gamma(x) \text{ is even}\}$ and $E_1 = \{x \in E \mid \gamma(x) \text{ is odd}\}$.

We have $E_0 \cap \langle f \rangle E_0 = \emptyset \notin \mu$ and hence $E_0 \notin \mu$.

Similarly $E_1 \notin \mu$.

Thus $E = E_0 \cup E_1 \notin \mu$. □

Lemma 58. f is the identity function on a set in μ.

Proof. We have shown $A, C, E \notin \mu$. But the points which lie in none of these sets are exactly points periodic with period 1 that is fixed points of f. Thus the set of fixed points of f belongs to the filter μ. □

3.1.2 The main theorem and its consequences

Theorem 59. For every atomic filter object a the morphism $((=)|_a; a; a)$ is the only

1. monovalued morphism of the category of reloids from a to a;
2. injective morphism of the category of reloids from a to a;
3. bijective morphism of the category of reloids from a to a.

Proof. We will prove only (1) because the rest follow from it.

Let f is a monovalued morphism from a to a. Then it exists a **Set**-morphism $(F; a; a)$ such that $F \in \operatorname{up} f$. Trivially $\langle \uparrow(F; a; a) \rangle a \supseteq a$ and thus $\langle F \rangle A \in \operatorname{up} a$ for every $A \in \operatorname{up} a$. Thus by the lemma we have that F is the identity function on a set in $\operatorname{up} a$ and so obviously f is an identity. □

Corollary 60. For every two atomic filter objects (with possibly different bases) $\mathcal{A}$ and $\mathcal{B}$ there exists at most one bijective reloid from $\mathcal{A}$ to $\mathcal{B}$.