Remark 39. The relation of being isomorphic for ultrafilters is traditionally called *Rudin-Keisler equivalence*.

Obvious 40. $(\geqslant_1) \supseteq (\geqslant_2)$.

Definition 41. Let Q and R are binary relations on the set of filter objects. I will denote $\mathbf{MonRld}_{Q,R}$ the directed multigraph with objects being filter objects and morphisms such monovalued reloids f that $(\operatorname{dom} f)\, Q\, \mathcal{A}$ and $(\operatorname{im} f)\, R\, \mathcal{B}$.

I will also denote $\mathbf{CoMonRld}_{Q,R}$ the directed multigraph with objects being filter objects and morphisms such injective reloids f that $(\operatorname{im} f)\, Q\, \mathcal{A}$ and $(\operatorname{dom} f)\, R\, \mathcal{B}$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

Theorem 42. For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_1 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
4. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
5. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
6. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
7. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

Proof.

(1)⇒(2). There exists a **Set**-morphism $f\colon \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$. We have

$$\operatorname{dom}\,(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A} \cap \operatorname{dom} f = \mathcal{A}$$

and

$$\operatorname{im}\,(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}\,(\mathsf{FCD})(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}\,(\uparrow f)|_{\mathcal{A}} = \langle \uparrow f \rangle \mathcal{A} \supseteq \mathcal{B}.$$

Thus $(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}}$ is a monovalued reloid such that $\operatorname{dom}\,(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and $\operatorname{im}\,(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} \supseteq \mathcal{B}$.

(2)⇒(3), (4)⇒(3), (5)⇒(6), (7)⇒(6). Obvious.

(3)⇒(1). We have $\mathcal{B} \subseteq \langle (\mathsf{FCD}) f \rangle \mathcal{A}$ for a monovalued reloid $f \in \mathsf{RLD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))$. Then there exists a **Set**-morphism $F\colon \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \subseteq \langle \uparrow F \rangle \mathcal{A}$ that is $\mathcal{A} \geqslant_1 \mathcal{B}$.

(6)⇒(7). $\operatorname{dom} f|_{\mathcal{B}} = \mathcal{B}$ and $\operatorname{im} f|_{\mathcal{B}} \subseteq \mathcal{A}$.

(2)⇔(5), (3)⇔(6), (4)⇔(7). By duality. □

Theorem 43. For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_2 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

Proof.

(1)⇒(2). Let $\mathcal{A} \geqslant_2 \mathcal{B}$ that is $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$ for some **Set**-morphism $f\colon \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$. Then $\operatorname{dom}\,(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and $\operatorname{im}\,(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}\,(\mathsf{FCD})(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}\,(\uparrow f)|_{\mathcal{A}} = \langle \uparrow f \rangle \mathcal{A} = \mathcal{B}$. So $(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}}$ is a sought for reloid.

(2)⇒(1). There exists a **Set**-morphism $F\colon \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $f = (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}$. Thus $\langle \uparrow F \rangle \mathcal{A} = \operatorname{im}(\uparrow F)|_{\mathcal{A}} = \operatorname{im}\,(\mathsf{FCD})(\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}} = \operatorname{im}\,(\mathsf{FCD}) f = \operatorname{im} f = \mathcal{B}$. Thus $\mathcal{A} \geqslant_2 \mathcal{B}$ is testified by the morphism F.

(2)⇔(3). By duality. □