Thus $g \circ f$ establishes a bijection which proves that $\mathcal{A}$ is isomorphic to $\mathcal{C}$. □

Lemma 35. Let $\operatorname{card} X = \operatorname{card} Y$, u is an atomic f.o. on X and v is an atomic f.o. on Y; let $A \in \operatorname{up} u$ and $B \in \operatorname{up} v$. Let $u \div A$ and $v \div B$ are directly isomorphic. Then if $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ we have u and v directly isomorphic.

Proof. Arbitrary extend the bijection witnessing being directly isomorphic to the sets $X \setminus A$ and $Y \setminus B$. □

Theorem 36. If $\operatorname{card} X = \operatorname{card} Y$ then being isomorphic and being directly isomorphic are the same for atomic f.o. u on X and v on Y.

Proof. That if two filter objects are isomorphic then they are directly isomorphic is obvious.

Let atomic f.o. u and v are isomorphic that is there is a bijection $f: A \to B$ where $A \in \operatorname{up} u$, $B \in \operatorname{up} v$ witnessing isomorphism of u and v.

If one of the filters u or v is a trivial atomic f.o. then the other is also a trivial atomic f.o. and as it is easy to show they are directly isomorphic. So we can assume u and v are not trivial atomic f.o.

If $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ our statement follows from the last lemma.

Now assume without loss of generality $\operatorname{card}(X \setminus A) < \operatorname{card}(Y \setminus B)$.

$\operatorname{card} B = \operatorname{card} Y$ because $\operatorname{card}(Y \setminus B) < \operatorname{card} Y$.

It is easy to show that thre exists $B' \supset B$ such that $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B')$ and $\operatorname{card} B' = \operatorname{card} B$.

We will find a bijection g from B to B' which witnesses direct isomorphism of v to v itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma u and v are directly isomorphic.

Let $D = B' \setminus B$. We have $D \notin \operatorname{up} v$.

There exists a set $E \subseteq B$ such that $\operatorname{card} E \geqslant \operatorname{card} D$ and $E \notin \operatorname{up} v$.

We have $\operatorname{card} E = \operatorname{card}(D \cup E)$ and thus there exists a bijection $h: E \to D \cup E$.

Let

$$g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\operatorname{im}(g|_{B \setminus E}) = B \setminus E$; $\operatorname{im}(g|_E) = \operatorname{im} h = D \cup E$;

$$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset.$$

Thus g is a bijection from B to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it's enough to show that $\langle g \rangle v = v$. Indeed it follows from $B \setminus E \in \operatorname{up} v$. □

Proposition 37.

1. For every $A \in \operatorname{up} \mathcal{A}$ and $B \in \operatorname{up} \mathcal{B}$ we have $\mathcal{A} \geqslant_2 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_2 \mathcal{B} \div B$.
2. For every $A \in \operatorname{up} \mathcal{A}$ and $B \in \operatorname{up} \mathcal{B}$ we have $\mathcal{A} \geqslant_1 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_1 \mathcal{B} \div B$.

Proof.

1. $\mathcal{A} \geqslant_2 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.
2. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$. □

Rudin-Keisler order of ultrafilters is considered in such a book as [6].

Proposition 38. For ultrafilters $\geqslant_2$ is the same as Rudin-Keisler ordering.

Proof. $x \geqslant_2 y$ iff there exist sets $A \in \operatorname{up} x$ and $B \in \operatorname{up} y$ a bijective **Set**-morphism $f: X \to Y$ such that $\operatorname{up}(y \div B) = \{C \in \mathscr{P} Y \mid \langle f^{-1} \rangle C \in \operatorname{up}(x \div A)\}$ that is when $C \in \operatorname{up}(y \div B) \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up}(x \div A)$ what is equivalent to $C \in \operatorname{up} y \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} x$ what is the definition of Rudin-Keisler ordering. □