Let now $C \in \operatorname{up} \langle \uparrow f \rangle \mathcal{A}$. Then $\uparrow^{\operatorname{Base}(\mathcal{A})} \langle f^{-1} \rangle C \supseteq \langle \uparrow f^{-1} \rangle \langle \uparrow f \rangle \mathcal{A} \supseteq \mathcal{A}$ and thus $\langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A}$. □

Corollary 20. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A}; \mathcal{B}) \Leftrightarrow \mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$ for every **Set**-morphism f from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$.

Corollary 21. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B}) \Leftrightarrow \mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$ for every **Set**-morphism f from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$.

Corollary 22. $\mathcal{A} \geqslant_2 \mathcal{B}$ iff it exists a **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$.

Corollary 23. $\operatorname{up} \mathcal{B} \supseteq f * \mathcal{A} \Leftrightarrow \mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$.

Corollary 24. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff it exists a **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$.

Proposition 25. For a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ the following are equivalent:

1. $\operatorname{up} \mathcal{B} = f * \mathcal{A}$.
2. $\forall C \in \operatorname{Base}(\mathcal{B}): (C \in \operatorname{up} \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A})$.
3. $\forall C \in \operatorname{Base}(\mathcal{A}): (\langle f \rangle C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A})$.
4. $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is a bijection from $\operatorname{up} \mathcal{A}$ to $\operatorname{up} \mathcal{B}$.
5. $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is a function onto $\operatorname{up} \mathcal{B}$.
6. $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$.
7. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$.
8. $f \in \operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{A}; \mathcal{B})$.

Proof.

(1)⇔(2). $\operatorname{up} \mathcal{B} = f * \mathcal{A} \Leftrightarrow \operatorname{up} \mathcal{B} = \{C \in \mathscr{P}\operatorname{Base}(\mathcal{B}) \mid \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A}\} \Leftrightarrow \forall C \in \operatorname{Base}(\mathcal{B}): (C \in \operatorname{up} \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A})$.

(2)⇔(3). Because f is a bijection.

(2)⇒(5). For every $C \in \operatorname{up} \mathcal{B}$ we have $\langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A}$ and thus $\langle f \rangle|_{\operatorname{up} \mathcal{A}} \langle f^{-1} \rangle C = \langle f \rangle \langle f^{-1} \rangle C = C$. Thus $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is onto $\operatorname{up} \mathcal{B}$.

(4)⇒(5). Obvious.

(5)⇒(4). We need to prove only that $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is an injection. But this follows from the fact that f is a bijection.

(4)⇒(3). We have $\forall C \in \operatorname{Base}(\mathcal{A}): ((\langle f \rangle|_{\operatorname{up} \mathcal{A}}) C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A})$ and consequently $\forall C \in \operatorname{Base}(\mathcal{A}): (\langle f \rangle C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A})$.

(6)⇔(1). From the last corollary.

(1)⇔(7). Obvious.

(7)⇔(8). Obvious. □

Corollary 26. The following are equivalent for every f.o. $\mathcal{A}$ and $\mathcal{B}$:

1. $\mathcal{A}$ is directly isomorphic to a f.o. $\mathcal{B}$.
2. There are a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{B})$
$$C \in \operatorname{up} \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A}$$
3. There are a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{B})$
$$\langle f \rangle C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A}.$$
4. There are a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is a bijection from $\operatorname{up} \mathcal{A}$ to $\operatorname{up} \mathcal{B}$.