

4. $\mathbf{CoMonRld}_{\subseteq, \supseteq}$;
5. $\mathbf{CoMonRld}_{\subseteq, =}$;
6. $\mathbf{CoMonRld}_{=, =}$.

Proof. We will prove only the first three. The rest follow from duality. We need to prove only that composition of morphisms is a morphism, because associativity and existence of identity morphism is evident. Using the theorem 22 we have:

1. Let $f \in \mathbf{Mor}_{\mathbf{MonRld}_{\subseteq, \supseteq}}(\mathcal{A}; \mathcal{B})$, $g \in \mathbf{Mor}_{\mathbf{MonRld}_{\subseteq, \supseteq}}(\mathcal{B}; \mathcal{C})$. Then $\text{dom } f \subseteq \mathcal{A}$, $\text{im } f \supseteq \mathcal{B}$, $\text{dom } g \subseteq \mathcal{B}$, $\text{im } g \supseteq \mathcal{C}$. So $\text{dom}(g \circ f) \subseteq \mathcal{A}$, $\text{im}(g \circ f) \supseteq \mathcal{C}$ that is $g \circ f \in \mathbf{Mor}_{\mathbf{MonRld}_{\subseteq, \supseteq}}(\mathcal{A}; \mathcal{C})$.
2. Let $f \in \mathbf{Mor}_{\mathbf{MonRld}_{\subseteq, =}}(\mathcal{A}; \mathcal{B})$, $g \in \mathbf{Mor}_{\mathbf{MonRld}_{\subseteq, =}}(\mathcal{B}; \mathcal{C})$. Then $\text{dom } f \subseteq \mathcal{A}$, $\text{im } f = \mathcal{B}$, $\text{dom } g \subseteq \mathcal{B}$, $\text{im } g = \mathcal{C}$. So $\text{dom}(g \circ f) \subseteq \mathcal{A}$, $\text{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathbf{Mor}_{\mathbf{MonRld}_{\subseteq, =}}(\mathcal{A}; \mathcal{C})$.
3. Let $f \in \mathbf{Mor}_{\mathbf{MonRld}_{=, =}}(\mathcal{A}; \mathcal{B})$, $g \in \mathbf{Mor}_{\mathbf{MonRld}_{=, =}}(\mathcal{B}; \mathcal{C})$. Then $\text{dom } f = \mathcal{A}$, $\text{im } f = \mathcal{B}$, $\text{dom } g = \mathcal{B}$, $\text{im } g = \mathcal{C}$. So $\text{dom}(g \circ f) = \mathcal{A}$, $\text{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathbf{Mor}_{\mathbf{MonRld}_{=, =}}(\mathcal{A}; \mathcal{C})$. \square

Definition 59. Let \mathbf{BijRld} is the groupoid of all bijections of the category of reloids. Its objects are filter objects and its morphisms from a f.o. \mathcal{A} to f.o. \mathcal{B} are monovalued injective trans-reloid f such that $\text{dom } f = \mathcal{A}$ and $\text{im } f = \mathcal{B}$.

Theorem 60. Filter objects \mathcal{A} and \mathcal{B} are isomorphic iff $\mathbf{Mor}_{\mathbf{BijRld}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

Proof.

\Rightarrow . Let \mathcal{A} and \mathcal{B} are isomorphic. Then there are sets $A \in \text{up } \mathcal{A}$, $B \in \text{up } \mathcal{B}$ and a bijective function $F: A \rightarrow B$ such that $\langle F \rangle: \mathcal{P}A \cap \text{up } \mathcal{A} \rightarrow \mathcal{P}B \cap \text{up } \mathcal{B}$ is a bijection.

Obviously $f = F|_{\mathcal{A}}^{\mathbf{RLD}(A; B)}$ is monovalued and injective.

$$\begin{aligned} \text{im } f &= \bigcap^{\mathfrak{F}} \left\{ \text{im } G \mid G \in F|_{\mathcal{A}}^{\mathbf{RLD}(A; B)} \right\} = \bigcap^{\mathfrak{F}} \left\{ \text{im}(H \cap F|_X) \mid H \in F|_{\mathcal{A}}^{\mathbf{RLD}(A; B)}, X \in \text{up } \mathcal{A} \right\} = \\ &= \bigcap^{\mathfrak{F}} \{ \text{im } F|_P \mid P \in \text{up } \mathcal{A} \} = \bigcap^{\mathfrak{F}} \{ \langle F \rangle P \mid P \in \text{up } \mathcal{A} \} = \bigcap^{\mathfrak{F}} \{ \langle F \rangle P \mid P \in \mathcal{P}A \cap \text{up } \mathcal{A} \} = \\ &= \bigcap^{\mathfrak{F}} (\mathcal{P}B \cap \text{up } \mathcal{B}) = \bigcap^{\mathfrak{F}} \text{up } \mathcal{B} = \mathcal{B}. \end{aligned}$$

Thus $\text{dom } f = \mathcal{A}$ and $\text{im } f = \mathcal{B}$.

\Leftarrow . Let f is a monovalued injective trans-reloid such that $\text{dom } f = \mathcal{A}$ and $\text{im } f = \mathcal{B}$. Then exist a function F' and an injective binary relation F'' such that $F', F'' \in \text{up } f$. Thus $F = F' \cap F''$ is an injection such that $F \in \text{up } f$. The function F is a bijection from $A = \text{dom } F$ to $B = \text{im } F$. The function $\langle F \rangle$ is an injection on $\mathcal{P}A \cap \text{up } \mathcal{A}$ (and moreover on $\mathcal{P}A$). Simple to show that $\forall X \in \mathcal{P}A \cap \text{up } \mathcal{A}: \langle F \rangle X \in \mathcal{P}B \cap \text{up } \mathcal{B}$ and similarly $\forall Y \in \mathcal{P}B \cap \text{up } \mathcal{B}: \langle F \rangle^{-1} Y = \langle F^{-1} \rangle Y \in \mathcal{P}A \cap \text{up } \mathcal{A}$. Thus $\langle F \rangle|_{\mathcal{P}A \cap \text{up } \mathcal{A}}$ is a bijection $\mathcal{P}A \cap \text{up } \mathcal{A} \rightarrow \mathcal{P}B \cap \text{up } \mathcal{B}$. So filter objects \mathcal{A} and \mathcal{B} are isomorphic. \square

Proposition 61. $(\geq_1) = (\subseteq) \circ (\geq_2)$ (when we limit to small f.o.).

Proof. $\mathcal{A} \geq_1 \mathcal{B}$ iff exists a function $f: \bigcup \text{up } \mathcal{A} \rightarrow \bigcup \text{up } \mathcal{B}$ such that $B \subseteq \langle f \rangle \mathcal{A}$. But $B \subseteq \langle f \rangle \mathcal{A}$ is equivalent to $\exists \mathcal{B}' \in \mathfrak{F}: (\mathcal{B} \subseteq \mathcal{B}' \wedge \mathcal{B}' = \langle f \rangle \mathcal{A})$. So $\mathcal{A} \geq_1 \mathcal{B}$ is equivalent to existence of $\mathcal{B}' \in \mathfrak{F}$ such that $\mathcal{B} \subseteq \mathcal{B}'$ and existence of a function $f: \bigcup \text{up } \mathcal{A} \rightarrow \bigcup \text{up } \mathcal{B}$ such that $\mathcal{B}' = \langle f \rangle \mathcal{A}$. That is equivalent to $\mathcal{A}((\subseteq) \circ (\geq_2)) \mathcal{B}$. \square

Proposition 62. If b is an atomic f.o. and $A \neq \emptyset$ then $b \geq_1 A \Leftrightarrow b \geq_2 A$. [TODO: Generalize it for f.o. on different base sets.]

Proof. We need to prove only $b \geq_1 A \Rightarrow b \geq_2 A$. If $b \geq_1 A$ then there exists a monovalued reloid f such that $\text{dom } f = b$ and $\text{im } f \supseteq A$. Then $\text{im } f = \text{im}(\mathbf{FCD})f \in \{\emptyset\} \cup \text{atoms}^{\mathfrak{F}}$ because $(\mathbf{FCD})f$ is a monovalued funcoid. So $\text{im } f = A$ (taken in account $A \neq \emptyset$) and thus $b \geq_2 A$. \square

Corollary 63. For atomic filter objects \geq_1 is the same as \geq_2 .

Thus I will write simply \geq for atomic f.o.