I will also denote $\mathbf{CoMonRld}_{Q,R}$ the directed multigraph with objects being filter objects and morphisms such injective trans-reloids $f$ that $(\operatorname{im} f)\, Q\, \mathcal{A}$ and $(\operatorname{dom} f)\, R\, \mathcal{B}$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

**Theorem 56.** For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_1 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
4. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
5. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
6. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
7. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

**Proof.**

**(1) ⇒ (2).** There exists a **Set**-morphism $f: \bigcup \operatorname{up} \mathcal{A} \to \bigcup \operatorname{up} \mathcal{B}$ such that $\mathcal{B} \subseteq \langle f \rangle \mathcal{A}$. We have

$$\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{RLD}} = \mathcal{A} \cap^{\mathfrak{F}} \operatorname{dom} f = \mathcal{A}$$

and

$$\operatorname{im} f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im} (\mathsf{FCD}) f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im} ((\mathsf{FCD}) f)|_{\mathcal{A}}^{\mathsf{FCD}} = \operatorname{im}\big(f|_{\mathcal{A}}^{\mathsf{FCD}}\big) = \langle f \rangle \mathcal{A} \supseteq \mathcal{B}.$$

Thus $f|_{\mathcal{A}}^{\mathsf{RLD}}$ is a monovalued reloid such that $\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{RLD}} = \mathcal{A}$ and $\operatorname{im} f|_{\mathcal{A}}^{\mathsf{RLD}} \supseteq \mathcal{B}$.

**(2) ⇒ (3), (4) ⇒ (3), (5) ⇒ (6), (7) ⇒ (6).** Obvious.

**(3) ⇒ (1).** We have $\mathcal{B} \subseteq \langle f \rangle \mathcal{A}$. Then there exists a **Set**-morphism $F: \bigcup \operatorname{up} \mathcal{A} \to \bigcup \operatorname{up} \mathcal{B}$ such that $\mathcal{B} \subseteq \langle F \rangle \mathcal{A}$ that is $\mathcal{A} \geqslant_1 \mathcal{B}$.

**(6) ⇒ (7).** $\operatorname{dom} f|_{\mathcal{B}} = \mathcal{B}$ and $\operatorname{im} f|_{\mathcal{B}} \subseteq \mathcal{A}$.

**(2) ⇔ (5), (3) ⇔ (6), (4) ⇔ (7).** By duality. □

**Theorem 57.** For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_2 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

**Proof.**

**(1) ⇒ (2).** Let $\mathcal{A} \geqslant_2 \mathcal{B}$ that is $\mathcal{B} = \langle f \rangle \mathcal{A}$ for some $f: \bigcup \operatorname{up} \mathcal{A} \to \bigcup \operatorname{up} \mathcal{B}$. Then $\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{RLD}} = \mathcal{A}$ (where $\mathsf{RLD}$ is taken on the set $\bigcup \operatorname{up} \mathcal{A} \cup \bigcup \operatorname{up} \mathcal{B}$) and $\operatorname{im} f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im} (\mathsf{FCD}) f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im} ((\mathsf{FCD}) f)|_{\mathcal{A}}^{\mathsf{FCD}} = \operatorname{im}\big(f|_{\mathcal{A}}^{\mathsf{FCD}}\big) = \langle f \rangle \mathcal{A} = \mathcal{B}$. So $f|_{\mathcal{A}}^{\mathsf{RLD}}$ is a sought for trans-reloid.

**(2) ⇒ (1).** There exists a function $F: \bigcup \operatorname{up} \mathcal{A} \to \bigcup \operatorname{up} \mathcal{B}$ such that $f = F|_{\mathcal{A}}^{\mathsf{RLD}}$ (where $\mathsf{RLD}$ is taken on the set $\bigcup \operatorname{up} \mathcal{A} \cup \bigcup \operatorname{up} \mathcal{B}$). Thus $\langle F \rangle \mathcal{A} = \operatorname{im}\big(F|_{\mathcal{A}}^{\mathsf{FCD}}\big) = \operatorname{im} (\mathsf{FCD}) F|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im} (\mathsf{FCD}) f = \operatorname{im} f = \mathcal{B}$. Thus $\mathcal{A} \geqslant_2 \mathcal{B}$ is testified by the function $F$.

**(2) ⇔ (3).** By duality. □

**Theorem 58.** The following are categories (with trans-reloid composition):

1. $\mathbf{MonRld}_{\subseteq,\supseteq}$;
2. $\mathbf{MonRld}_{\subseteq,=}$;
3. $\mathbf{MonRld}_{=,=}$.