Proposition 39. For a bijective **Set**-morphism $f: \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ the following are equivalent:

1. $\operatorname{up}\mathcal{B} = f{*}\mathcal{A}$.
2. $\forall C \in \bigcup \operatorname{up}\mathcal{B}: \left(C \in \operatorname{up}\mathcal{B} \Leftrightarrow \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}\right)$.
3. $\forall C \in \bigcup \operatorname{up}\mathcal{B}: (\langle f\rangle C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A})$.
4. $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a bijection from $\operatorname{up}\mathcal{A}$ to $\operatorname{up}\mathcal{B}$.
5. $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a function onto $\operatorname{up}\mathcal{B}$.
6. $\mathcal{B} = \langle f\rangle\mathcal{A}$.
7. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$.
8. $f \in \operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{A}; \mathcal{B})$.

Proof.

(1) ⇔ (2). $\operatorname{up}\mathcal{B} = f{*}\mathcal{A} \Leftrightarrow \operatorname{up}\mathcal{B} = \left\{C \in \mathscr{P} \bigcup \operatorname{up}\mathcal{B} \mid \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}\right\} \Leftrightarrow \forall C \in \bigcup \operatorname{up}\mathcal{B}: \left(C \in \operatorname{up}\mathcal{B} \Leftrightarrow \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}\right)$.

(2) ⇔ (3). Because f is a bijection.

(2) ⇒ (5). For every $C \in \operatorname{up}\mathcal{B}$ we have $\left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}$ and thus $\langle f\rangle|_{\operatorname{up}\mathcal{A}}\left\langle f^{-1}\right\rangle C = \langle f\rangle\left\langle f^{-1}\right\rangle C = C \in \operatorname{up}\mathcal{B}$. Thus $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is onto $\operatorname{up}\mathcal{B}$.

(4) ⇒ (5). Obvious.

(5) ⇒ (4). We need to prove only that $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is an injection. But this follows from the fact that f is a bijection.

(4) ⇒ (3). We have $\forall C \in \bigcup \operatorname{up}\mathcal{B}: ((\langle f\rangle|_{\operatorname{up}\mathcal{A}})C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A})$ and consequently $\forall C \in \bigcup \operatorname{up}\mathcal{B}: (\langle f\rangle C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A})$.

(6) ⇔ (1). From the last corollary.

(1) ⇔ (7). Obvious.

(7) ⇔ (8). Obvious. □

Corollary 40. The following are equivalent for every f.o. $\mathcal{A}$ and $\mathcal{B}$:

1. $\mathcal{A}$ is directly isomorphic to a f.o. $\mathcal{B}$.
2. There are a bijective **Set**-morphism $f: \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that for every $C \in \mathscr{P} \bigcup \operatorname{up}\mathcal{B}$
$$C \in \operatorname{up}\mathcal{B} \Leftrightarrow \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}$$
3. There are a bijective **Set**-morphism $f: \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that for every $C \in \mathscr{P} \bigcup \operatorname{up}\mathcal{B}$
$$\langle f\rangle C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A}.$$
4. There are a bijective **Set**-morphism $f: \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a bijection from $\operatorname{up}\mathcal{A}$ to $\operatorname{up}\mathcal{B}$.
5. There are a bijective **Set**-morphism $f: \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a function onto $\operatorname{up}\mathcal{B}$.
6. There are a bijective **Set**-morphism $f: \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\mathcal{B} = \langle f\rangle\mathcal{A}$.
7. There are a bijective morphism $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$.

Proposition 41. $\mathbf{GreFunc}_1$ and $\mathbf{GreFunc}_2$ with function composition are categories.

Proof. Let $f: \mathcal{A} \to \mathcal{B}$ and $g: \mathcal{B} \to \mathcal{C}$ are morphisms of $\mathbf{GreFunc}_1$. Then $\mathcal{B} \subseteq \langle f\rangle\mathcal{A}$ and $\mathcal{C} \subseteq \langle g\rangle\mathcal{B}$. So $\langle g \circ f\rangle\mathcal{A} = \langle g\rangle\langle f\rangle\mathcal{A} \supseteq \langle g\rangle\mathcal{B} \supseteq \mathcal{C}$. Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_1$. Associativity law is evident. $\operatorname{Id}_{\bigcup \operatorname{up}\mathcal{A}}$ is the identity morphism of $\mathbf{GreFunc}_1$ for every f.o. $\mathcal{A}$.