Transitivity. Let $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ for some small f.o. $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then exist a set X such that $X \in \operatorname{up} \mathcal{A}$ and $X \in \operatorname{up} \mathcal{B}$ and $\mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} X \cap \operatorname{up} \mathcal{B}$ and a set Y such that $Y \in \operatorname{up} \mathcal{B}$ and $Y \in \operatorname{up} \mathcal{C}$ and $\mathscr{P} Y \cap \operatorname{up} \mathcal{B} = \mathscr{P} Y \cap \operatorname{up} \mathcal{C}$. So $X \cap Y \in \operatorname{up} \mathcal{A}$ because

$$\mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{B} = \mathscr{P}(X \cap Y) \cap \operatorname{up} \mathcal{B} \supseteq \{X \cap Y\} \cap \operatorname{up} \mathcal{B} \ni X \cap Y.$$

Similarly we have $X \cap Y \in \operatorname{up} \mathcal{C}$. Finally $\mathscr{P}(X \cap Y) \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{B} = \mathscr{P} X \cap \mathscr{P} Y \cap \operatorname{up} \mathcal{B} = \mathscr{P} X \cap \mathscr{P} Y \cap \operatorname{up} \mathcal{C} = \mathscr{P}(X \cap Y) \cap \operatorname{up} \mathcal{C}$. □

Definition 4. The *rebase* $\mathcal{A} \div A$ for a f.o. $\mathcal{A}$ and a set A (base) such that $\exists X \in \operatorname{up} \mathcal{A}: X \subseteq A$ is defined by the formula

$$\mathcal{A} \div A = \operatorname{up}^{-1}\{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$$

where "up" is taken for the set of f.o. on A.

Proposition 5. If $\exists X \in \operatorname{up} \mathcal{A}: X \subseteq A$ then:

1. $\mathcal{A} \div A$ is a f.o.
2. $\mathcal{A} \div A \sim \mathcal{A}$.

Proof.

1. We need to prove that $\{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$ is a filter. That it is an upper set is obvious. It is non-empty because $\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq A$ and thus $A \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$. Let $P, Q \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$. Then $P, Q \subseteq A$ and $\exists P' \in \operatorname{up} \mathcal{A}: P' \subseteq P$ and $\exists Q' \in \operatorname{up} \mathcal{A}: Q' \subseteq P$. So $P \cap Q \subseteq A$ and $P' \cap Q' \subseteq P \cap Q$. Thus $P \cap Q \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$.
2. $\mathscr{P}(A \cap \bigcup \operatorname{up} \mathcal{A}) \cap \operatorname{up} \mathcal{A} = \operatorname{up}(\mathcal{A} \div (A \cap \bigcup \operatorname{up} \mathcal{A})) = \{X \in \mathscr{P}(A \cap \bigcup \operatorname{up} \mathcal{A}) \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\} = \mathscr{P}(A \cap \bigcup \operatorname{up} \mathcal{A}) \cap \operatorname{up} \mathcal{A} = \mathscr{P} A \cap \operatorname{up} \mathcal{A} = \{X \in \mathscr{P} A \mid X \in \operatorname{up} \mathcal{A}\} = \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X, X \subseteq \bigcup \operatorname{up} \mathcal{A}\} = \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X, X \subseteq A \cap \bigcup \operatorname{up} \mathcal{A}\} = \mathscr{P}(A \cap \bigcup \operatorname{up} \mathcal{A}) \cap \operatorname{up}(\mathcal{A} \div A)$.

   Thus $\mathcal{A} \div A \sim \mathcal{A}$ because $A \cap \bigcup \operatorname{up} \mathcal{A} \supseteq X \cap \bigcup \operatorname{up} \mathcal{A} = X \in \operatorname{up} \mathcal{A}$ for some $X \in \operatorname{up} \mathcal{A}$ and $A \cap \bigcup \operatorname{up} \mathcal{A} \supseteq X \cap \bigcup \operatorname{up} \mathcal{A} \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\} = \operatorname{up}(\mathcal{A} \div A)$. □

Proposition 6. $A \in \operatorname{up} \mathcal{A} \Rightarrow \operatorname{up}(\mathcal{A} \div A) = \mathscr{P} A \cap \operatorname{up} \mathcal{A}$.

Proof. Let $A \in \operatorname{up} \mathcal{A}$. Then $\operatorname{up}(\mathcal{A} \div A) = \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\} = \{X \in \mathscr{P} A \mid X \in \operatorname{up} \mathcal{A}\} = \mathscr{P} A \cap \operatorname{up} \mathcal{A}$. □

Lemma 7. If $\mathcal{A} \sim \mathcal{B}$ then $\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Leftrightarrow \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X$ for every f.o. $\mathcal{A}$, $\mathcal{B}$, and a set X.

Proof. We will prove $\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Rightarrow \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X$ (the other direction is similar).

We have $\mathscr{P} K \cap \operatorname{up} \mathcal{A} = \mathscr{P} K \cap \operatorname{up} \mathcal{B}$ for some set K.

$\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Rightarrow \exists Y \in \mathscr{P} K \cap \operatorname{up} \mathcal{A}: Y \subseteq X \Rightarrow \exists Y \in \mathscr{P} K \cap \operatorname{up} \mathcal{B}: Y \subseteq X \Rightarrow \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X$. □

Proposition 8. If $\mathcal{A} \sim \mathcal{B}$ then $\mathcal{B} = \mathcal{A} \div \bigcup \operatorname{up} \mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. $\mathscr{P} Y \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \operatorname{up} \mathcal{B}$ for some set $Y \in \operatorname{up} \mathcal{A}$, $Y \in \operatorname{up} \mathcal{B}$. Thus exists a set $X \in \operatorname{up} \mathcal{A}$ such that $X \in \operatorname{up} \mathcal{B} \subseteq \bigcup \operatorname{up} \mathcal{B}$. Thus $\exists X \in \operatorname{up} \mathcal{A}: X \subseteq \bigcup \operatorname{up} \mathcal{B}$ and so $\mathcal{A} \div \bigcup \operatorname{up} \mathcal{B}$ is a properly defined f.o.

$X \in \operatorname{up}(\mathcal{A} \div \bigcup \operatorname{up} \mathcal{B}) \Leftrightarrow X \in \mathscr{P} \bigcup \operatorname{up} \mathcal{B} \wedge \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Leftrightarrow X \in \mathscr{P} \bigcup \operatorname{up} \mathcal{B} \wedge \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X \Leftrightarrow X \in \operatorname{up} \mathcal{B}$ (the lemma used). □

2.2 Trans-reloids

Definition 9. A *trans-reloid* is a f.o. on the set $A \times B$ for some small sets A and B.

I will denote the set of trans-reloids for sets A, B as $\mathsf{RLD}(A; B)$.

Definition 10. $\operatorname{Src} f = \bigcup \operatorname{Pr}_0 \operatorname{up} f$; $\operatorname{Dst} f = \bigcup \operatorname{Pr}_1 \operatorname{up} f$ for every trans-reloid f.