2. injective;
3. entirely defined if $A \subseteq B$;
4. surjective if $B \subseteq A$.

Proof.

1. $(A \rightleftarrows B) \circ (B \rightleftarrows A) \sqsubseteq 1_B^{\mathbf{Rel}}$; $(A \rightleftarrows B) \circ (A \rightleftarrows B)^{-1} \sqsubseteq 1_B^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B) \circ (A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \sqsubseteq 1_B^{\mathcal{C}}$.
2. $(B \rightleftarrows A) \circ (A \rightleftarrows B) \sqsubseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows B)^{-1} \circ (A \rightleftarrows B) \sqsubseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \circ (A \rightleftarrows^{\mathcal{C}} B) \sqsubseteq 1_A^{\mathcal{C}}$.
3. $(B \rightleftarrows A) \circ (A \rightleftarrows B) \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows B)^{-1} \circ (A \rightleftarrows B) \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \circ (A \rightleftarrows^{\mathcal{C}} B) \sqsupseteq 1_A^{\mathcal{C}}$.
4. $(A \rightleftarrows B) \circ (B \rightleftarrows A) \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows B) \circ (A \rightleftarrows B)^{-1} \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B) \circ (A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \sqsupseteq 1_A^{\mathcal{C}}$. □

4 Rectangular embedding-restriction

Definition 25. $\iota_{B_0, B_1} f = (A_1 \rightleftarrows^{\mathcal{C}} B_1) \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0)$ for $f \in \operatorname{Mor}_{\mathcal{C}}(A_0; A_1)$.

For brevity $\iota_B f = \iota_{B,B} f$.

Proposition 26. $\iota_{\operatorname{Src} f, \operatorname{Dst} f} f = f$.

Proof. $\iota_{\operatorname{Src} f, \operatorname{Dst} f} f = (\operatorname{Dst} f \rightleftarrows^{\mathcal{C}} \operatorname{Dst} f) \circ f \circ (\operatorname{Src} f \rightleftarrows^{\mathcal{C}} \operatorname{Src} f) = 1_{\mathcal{C}}^{\operatorname{Dst} f} \circ f \circ 1_{\mathcal{C}}^{\operatorname{Src} f} = f$. □

Proposition 27. The function $\iota_{B_0, B_1}|_{f \in \operatorname{Mor}_{\mathcal{C}}(A_0; A_1)}$ is injective, if $A_0 \subseteq B_0 \wedge A_1 \subseteq B_1$.

Proof. Because $A_1 \rightleftarrows^{\mathcal{C}} B_1$ is a monomorphism and $A_0 \rightleftarrows^{\mathcal{C}} B_0$ is an epimorphism. □

Proposition 28. $\iota_{C_0, C_1} \iota_{B_0, B_1} f = \iota_{C_0, C_1} f$ for $B_0 \supseteq A_0 \cap C_0$, $B_1 \supseteq A_1 \cap C_1$ and $f: A_0 \rightarrow A_1$.

Proof. $\iota_{C_0, C_1} \iota_{B_0, B_1} f = (B_1 \rightleftarrows^{\mathcal{C}} C_1) \circ (A_1 \rightleftarrows^{\mathcal{C}} B_1) \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) \circ (C_0 \rightleftarrows^{\mathcal{C}} B_0) = (A_1 \rightleftarrows^{\mathcal{C}} C_1) \circ f \circ (C_0 \rightleftarrows^{\mathcal{C}} A_0) = \iota_{C_0, C_1} f$. □

Proposition 29. Let $f: A_0 \rightarrow A_1$ and $g: A_1 \rightarrow A_2$ and $A_1 \subseteq B_1$. Then $\iota_{B_0, B_2}(g \circ f) = \iota_{B_1, B_1} g \circ \iota_{B_0, B_1} f$.

Proof. $\iota_{B_0, B_2}(g \circ f) = (A_2 \rightleftarrows^{\mathcal{C}} B_2) \circ (g \circ f) \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) = (A_2 \rightleftarrows^{\mathcal{C}} B_2) \circ g \circ \operatorname{id}_{A_1} \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) = (A_2 \rightleftarrows^{\mathcal{C}} B_2) \circ g \circ (B_1 \rightleftarrows A_1) \circ (A_1 \rightleftarrows B_1) \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) = \iota_{B_1, B_1} g \circ \iota_{B_0, B_1} f$. □

5 Examples of partially ordered dagger categories under Rel

5.1 Generalized rebase of filters

In [2] I defined *rebase* $\mathcal{A} \div A$ for a set-theoretic filter $\mathcal{A}$ and a set X such that $\exists X \in \mathcal{A}: X \subseteq A$.

Now define a generalized rebase for every set-theoretic filter $\mathcal{A}$ and every set A:

Definition 30. $\mathcal{A} \div A = \bigsqcap \{\uparrow^A (X \cap A) \mid X \in \mathcal{A}\}$.

Proposition 31. In the case of $\exists X \in \mathcal{A}: X \subseteq A$ these two definitions coincide.

Proof. Let $\exists X \in \mathcal{A}: X \subseteq A$. Then as it is proved in [2] $\{X \in \mathscr{P} A \mid \exists Y \in \mathcal{A}: Y \subseteq X\}$ is a filter.

If $P \in \{X \in \mathscr{P} A \mid \exists Y \in \mathcal{A}: Y \subseteq X\}$ then $P \in \mathscr{P} A$ and $Y \subseteq P$ for some $Y \in \mathcal{A}$. Thus $P \supseteq Y \cap A \in \bigsqcap \{\uparrow^B (Y \cap A) \mid Y \in \mathcal{A}\}$.

If $P \in \bigsqcap \{\uparrow^B (X \cap A) \mid X \in \mathcal{A}\}$ then by properties of generalized filter bases, there exists $X \in \mathcal{A}$ such that $P \supseteq X \cap A$. Also $P \in \mathscr{P} A$. Thus $P \supseteq X$. Thus $P \in \{X \in \mathscr{P} A \mid \exists Y \in \mathcal{A}: Y \subseteq X\}$.

[TODO: Clear this proof: wording, consistent use of letters.] □