**Obvious 10.** $A \rightleftarrows A = 1^A_{\mathbf{Rel}}$.

**Obvious 11.** $(A \rightleftarrows B)^{-1} = B \rightleftarrows A$.

**Definition 12.** *Dagger functor* between two dagger categories is a functor between these categories, which commutes with the daggers. [TODO: Clearer wording.]

**Definition 13.** *Category under* **Rel** is a pair $(C; \uparrow)$ where $C$ is a category whose objects are small sets and $\uparrow$ is an identity-on-objects functor $\mathbf{Rel} \rightarrow C$. I call $\uparrow$ *up-arrow functor*. [TODO: $A \sqsubseteq B \rightarrow A \subseteq B$ for sets.]

**Definition 14.** *Dagger category under* **Rel** is a pair $(C; \uparrow)$ where $C$ is a dagger category whose objects are small sets and $\uparrow$ is a dagger identity-on-objects functor $\mathbf{Rel} \rightarrow C$.

**Definition 15.** $A \rightleftarrows^{\mathcal{C}} B = \uparrow (A \rightleftarrows B)$. In other words, $\rightleftarrows^{\mathcal{C}} = \uparrow \circ \rightleftarrows$.

**Proposition 16.** $A \rightleftarrows^{\mathcal{C}} A = 1^A_{\mathcal{C}}$.

**Proof.** $A \rightleftarrows^{\mathcal{C}} A = \uparrow(A \rightleftarrows A) = \uparrow 1_{\mathbf{Rel}} = 1^A_{\mathcal{C}}$. □

**Proposition 17.** If $f: X \rightarrow Y$ is a **Rel**-morphism and $\operatorname{im} f = A \subseteq Y$ then

$$(A \rightleftarrows Y) \circ (Y \rightleftarrows A) \circ f = f.$$

**Proof.** $(A \rightleftarrows Y) \circ (Y \rightleftarrows A) \circ f = \mathrm{id}_A \circ f = f$. □

**Corollary 18.** If $f: X \rightarrow Y$ is a morphism of a category under **Rel** and $\operatorname{im} f = A \subseteq Y$, then

$$(A \rightleftarrows^{\mathcal{C}} Y) \circ (Y \rightleftarrows^{\mathcal{C}} A) \circ \uparrow f = \uparrow f.$$

**Proposition 19.**

1. If $A \subseteq B$ then $A \rightleftarrows^{\mathcal{C}} B$ is a monomorphism.
2. If $A \supseteq B$ then $A \rightleftarrows^{\mathcal{C}} B$ is a epimorphism.

**Proof.** We'll prove only the first as the second is dual.

Let $(A \rightleftarrows^{\mathcal{C}} B) \circ f = (A \rightleftarrows^{\mathcal{C}} B) \circ g$. Then $(B \rightleftarrows^{\mathcal{C}} A) \circ (A \rightleftarrows^{\mathcal{C}} B) \circ f = (B \rightleftarrows^{\mathcal{C}} A) \circ (A \rightleftarrows^{\mathcal{C}} B) \circ g$; $1^A \circ f = 1^A \circ g$; $f = g$. □

**Proposition 20.** $(B \rightleftarrows C) \circ (A \rightleftarrows B) = A \rightleftarrows C$ iff $B \supseteq A \cap C$ (for every sets $A$, $B$, $C$).

**Proof.** $(B \rightleftarrows C) \circ (A \rightleftarrows B) = A \rightleftarrows C$ is equivalent to:
$(B; C; \mathrm{id}_{B \cap C}) \circ (A; B; \mathrm{id}_{A \cap B}) = (A; C; \mathrm{id}_{A \cap C})$;
$(A; C; \mathrm{id}_{A \cap B \cap C}) = (A; C; \mathrm{id}_{A \cap C})$;
$A \cap B \cap C = A \cap C$;
$B \supseteq A \cap C$. □

**Corollary 21.** $(B \rightleftarrows^{\mathcal{C}} C) \circ (A \rightleftarrows^{\mathcal{C}} B) = (A \rightleftarrows^{\mathcal{C}} C)$ if $B \supseteq A \cap C$ (for every sets $A$, $B$, $C$).

**Definition 22.** *Partially ordered dagger category under* **Rel** is a category which is both a partially ordered dagger category and a category under **Rel** such that $\uparrow \circ f^{-1} = (\uparrow \circ f)^{\dagger}$ and $A \sqsubseteq B \Rightarrow \uparrow A \sqsubseteq \uparrow B$.

**Proposition 23.** $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} = B \rightleftarrows^{\mathcal{C}} A$ for a dagger category under **Rel**.

**Proof.** $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} = (\uparrow(A \rightleftarrows B))^{\dagger} = \uparrow(A \rightleftarrows B)^{-1} = \uparrow(B \rightleftarrows A) = B \rightleftarrows^{\mathcal{C}} A$. □

**Proposition 24.** For a partially ordered dagger category $\mathcal{C}$ under **Rel** we have $A \rightleftarrows^{\mathcal{C}} B$ is:

1. monovalued;