Equalizers and co-Equalizers in Certain Categories

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1 Draft status

It is a rough draft. Errors are possible. Subscribe to my blog for further results.

http://math.stackexchange.com/questions/540220/right-adjoint-of-forgetful-functor-from-top

[TODO: Change notation $\prod \to \prod^{(L)}$.]

2 Categories with embeddings

Note 1. This section in not used below, it is just to feed your intuition.

The following generalizes the well known concept of embedding function $A \hookrightarrow B$ for from a set A to a set B where $A \subseteq B$.

I will set that the unique morphism from an object A to an object B of a thin category is equal to the pair $(A; B)$.

Definition 2. A *category with embeddings of objects* is a dagger category with a preorder of the set of objects together with a functor $\hookrightarrow$ (we will denote applying this functor to the object $(A; B)$ as $A \hookrightarrow B$.) such that:

- $\hookrightarrow$ is an identity on objects.
- Every $A \hookrightarrow B$ is a monomorphism.
- $(A \hookrightarrow B)^{\dagger} \circ (A \hookrightarrow B) = 1_A$.

Obvious 3. $A \hookrightarrow B$ is defined when $(A; B)$ is a morphism of the preorder that is when $A \sqsubseteq B$.

Obvious 4. $A \hookrightarrow B: A \to B$ when $A \sqsubseteq B$.

Proposition 5. $A \hookrightarrow A = 1_A$.

Proof. Because $(A; A)$ is an identity morphism and $\hookrightarrow$ preserves identities. □

Proposition 6. $(B \hookrightarrow C) \circ (A \hookrightarrow B) = A \hookrightarrow C$ whenever $A \sqsubseteq B \sqsubseteq C$.

Proof. $(B \hookrightarrow C) \circ (A \hookrightarrow B) = \hookrightarrow(B; C) \circ \hookrightarrow(A; B) = \hookrightarrow((B; C) \circ (A; B)) = \hookrightarrow(A; C) = A \hookrightarrow C$. □

3 Categories under Rel

Definition 7. The **Rel**-morphism $A \rightleftarrows B$ (*restriction-embedding*) is defined by the formula: $A \rightleftarrows B = (A; B; \mathrm{id}_{A \cap B})$.

Obvious 8. If $A \subseteq B$ then $A \rightleftarrows B$ is an embedding $A \hookrightarrow B = (A; B; \mathrm{id}_A)$.

Obvious 9. If $A \supseteq B$ then $A \rightleftarrows B = (A; B; \mathrm{id}_B)$.