5.2.1. Connectedness of unions of sets

Lemma 3 *If $X \cup Y = A \cup B$ and $X, Y \neq \emptyset$ and $X \cap Y = \emptyset$ then either $\{X, Y\} = \{A, B\}$ or A intersects both X and Y or B intersects both X and Y (for every sets A, B, X, Y).*

Proof Let $\{X, Y\} \neq \{A, B\}$. Suppose that "A intersects both X and Y" does not hold (for example suppose that $A \cap X = 0$) and prove "B intersects both X and Y".

We have $X \subseteq B$ and thus $B \cap X \neq 0$. If also $B \cap Y = 0$ then $B \subseteq X$. So $X = B$ and thus either $Y = A$ what contradicts to our supposition or $A \supset Y$ in which case A intersects both X and Y. □

Theorem 7 *If sets $A, B \in \mathscr{P}U$ are connected regarding an extendable connector space $(U; r)$ and $A\ r\ B$ then $A \cup B$ is also connected regarding $(U; r)$.*

Proof We need to prove that

$$\forall X, Y \in \mathscr{P}(A \cup B) \setminus \{\emptyset\} : (X \cup Y = A \cup B \wedge X \cap Y = \emptyset \Rightarrow X\ r\ Y).$$

Let $X, Y \in \mathscr{P}(A \cup B) \setminus \{\emptyset\}$ and $X \cup Y = A \cup B \wedge X \cap Y = \emptyset$. Then by the lemma either $\{X, Y\} = \{A, B\}$ and thus $X\ r\ Y \Leftrightarrow A\ r\ B$ so having $X\ r\ Y$, or A intersects both X and Y or B intersects both X and Y. Consider for example then case $X \cap A \neq \emptyset$ and $Y \cap A \neq \emptyset$.

In this case we have $(X \cap A) \cup (Y \cap A) = (X \cup Y) \cap A = (A \cup B) \cap A = A$ and $(X \cap A) \cap (Y \cap A) \subseteq X \cap Y = \emptyset$. Thus $X \cap A\ r\ Y \cap A$ and consequently $X\ r\ Y$ (taken in account extendability). □

Corollary 3 *If sets $A, B \in \mathscr{P}U$ are connected regarding an extendable connector space $(U; r)$ and $A \cap B \neq \emptyset$ then $A \cup B$ is also connected regarding $(U; r)$.*

Proof Replace r with its normalization $N(r)$. This preserves the same connectedness. $A \cap B \neq \emptyset \Rightarrow A\ N(r)\ B$. Thus we can apply the theorem. □

There holds also infinite version of the previous corollary:

Theorem 8 *If $S \in \mathscr{P}\mathscr{P}U$ is a collection of connected (regarding an extendable connector space $(U; r)$) sets and $\bigcap S \neq \emptyset$ then $\bigcup S$ is connected (regarding this connector space).*

Proof Let $\{X, Y\}$ is a partition of $\bigcup S$. Then exist a point $p \in \bigcap S$ such that $p \in X$ or $p \in Y$. Without lost of generality we may assume $p \in X$. Since $Y \neq \emptyset$, we have $q \in Y$ for some $q \in \bigcup S$ that is $q \in A$ for some $A \in S$. So $A \cap X, A \cap Y \neq \emptyset$ and thus $\{A \cap X, A \cap Y\}$ is a partition of A. Since A is connected, we have $A \cap X\ r\ A \cap Y$ and thus (taken in account extendability) $X\ r\ Y$. So $\bigcup S$ is connected. □