4.6.4. Connectors for uniform connectedness

Let's find a connector which generates the same connectedness as the described above uniform connectedness.

Proposition 10 $\forall x \in U : [\{x\} \times^C \{x\}) \supseteq S^*(\mu)$ *for every generalized uniform space* $\mu = (U; f)$.

Proof $S^*(\mu) = \bigcup^{\mathcal{U}} \{[S(f)) \mid f \in \mu\}$. But $\{x\} \times^C \{x\} \subseteq S(f)$; thus $[\{x\} \times^C \{x\}) \supseteq [S(f))$ and consequently $\bigcup^{\mathcal{U}} \{[S(f)) \mid f \in \mu\} \subseteq [\{x\} \times^C \{x\})$. □

Lemma 2 $[\bigcup S) \supseteq F \Leftrightarrow \forall X \in S : [X) \supseteq F$ *for every collection* S *of sets and every filter* F.

Proof

$\Rightarrow$ Obvious.

$\Leftarrow$ Let $\forall X \in S : [X) \supseteq F$ that is $\forall X \in S, Y \in F : X \subseteq Y$. Then $\forall Y \in F : \bigcup S \subseteq Y$ that is $[\bigcup S) \supseteq F$.

□

From the above lemma follows that

$$[A \times^C A) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \Leftrightarrow$$
$$\forall x \in A : [\{x\} \times^C (A \setminus \{x\})) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \wedge [\{x\} \times^C \{x\}) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)).$$

Because $\forall x \in A : [\{x\} \times^C \{x\}) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A))$, we have

$$[A \times^C A) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \Leftrightarrow \forall x \in A : [\{x\} \times^C (A \setminus \{x\})) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times A))$$

Consequently

$$[A \times^C A) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \Leftrightarrow$$
$$\forall X, Y \in \mathscr{P} A : (X \cap Y = \emptyset \wedge X \cup Y = A \Rightarrow [X \times^C Y) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)).$$

So, our sought-for connector is defined (for example) by the formula

$$X \; r \; Y \Leftrightarrow [X \times^C Y) \supseteq S^*(\mu \cup^{\mathcal{U}} [(X \cup Y) \times^C (X \cup Y))).$$

A is connected regarding μ iff $\forall f \in \mu, X, Y \in \mathscr{P} U : (X \cup Y = A \Rightarrow X \, [f] \, Y) \Leftrightarrow \forall X, Y \in \mathscr{P} U : (X \cup Y = A \Rightarrow \forall f \in \mu : X \, [f] \, Y)$. Thus

$$X \; r \; Y \Leftrightarrow \forall f \in \mu : X \, [f] \, Y \Leftrightarrow \forall f \in \mu : X \times Y \cap f \neq \emptyset \tag{4}$$

is also a connector which induces uniform connectedness.

If μ is a uniformity, $X \; r \; Y \Leftrightarrow X \; \delta \; Y$ where δ is the proximity induced by μ. Thus my definition of uniform connectedness is equivalent to traditional definition of uniform connectedness. (See theorem 1 in [3].)