Proof

(1)⇒(2) Let for every $a, b \in A$ there is a path between a and b in A through μ. Then $a\ (S(\mu \cap^C (A \times A)))\ b$ for every $a, b \in A$. It is possible only when $S(\mu \cap^C (A \times A)) \supseteq A \times A$.

(3)⇒(1) For every two vertices a and b we have $a\ (S(\mu \cap (A \times^C A)))\ b$. So (by the previous theorem) for every two vertices a and b exist path from a to b.

(3)⇒(4) Suppose that $\neg(X \left[\mu \cap (A \times^C A)\right] Y)$ for some $X, Y \in \mathscr{P}U \setminus \{\emptyset\}$ such that $X \cup Y = A$ and $X \cap Y = \emptyset$. Then by a lemma $\neg(X \left[(\mu \cap (A \times^C A))^n\right] Y)$ for every $n \in \mathbb{N}$. Consequently $\neg(X \left[S(\mu \cap (A \times^C A))\right] Y)$. So $S(\mu \cap (A \times^C A)) \neq A \times A$.

(4)⇒(3) If $\left\langle S(\mu \cap^C (A \times A))\right\rangle \{v\} = A$ for every vertex v then $S(\mu \cap^C (A \times A)) = A \times^C A$. Consider the remaining case when $V \overset{\text{def}}{=} \left\langle S(\mu \cap (A \times^C A))\right\rangle \{v\} \subset A$ for some vertex v. Let $W = A \setminus V$. If $\operatorname{card} A = 1$ then $S(\mu \cap^C (A \times A)) \supseteq (=)|_A = A \times^C A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V\,[\mu]\,W$ what is equivalent to $V \left[\mu \cap^C (A \times A)\right] W$ that is $\left\langle \mu \cap^C (A \times A)\right\rangle V \cap W \neq \emptyset$. This is impossible because $\left\langle \mu \cap (A \times^C A)\right\rangle V = \left\langle \mu \cap (A \times^C A)\right\rangle \left\langle S(\mu \cap (A \times^C A))\right\rangle V \subseteq \left\langle S(\mu \cap (A \times^C A))\right\rangle V = V$.

(2)⇒(3) Because $S(\mu \cap (A \times^C A)) \subseteq A \times^C A$.

(5)⇒(4) Obvious.

(4)⇒(5) Let (4) holds and let $X \cup Y = A$. If $X = Y = A$ then $X\,[\mu]\,Y$ because $A \neq \emptyset$. Otherwise $X \subset A$ or $Y \subset A$. Let for example $X \subset A$. Then $Y \setminus X \neq \emptyset$. So $X\,[\mu]\,Y \setminus X$ by (4) and consequently $X\,[\mu]\,Y$.

□

Corollary 2 *A set A is connected regarding a digraph μ iff it is connected regarding $\mu \cap (A \times^C A)$.*

Theorem 4 *The following statements are equivalent for each digraph $\mu = (U; f)$ and sets $X, Y \in \mathscr{P}U$:*

1. $X\ T(U; \tau)\ Y$;

2. $X \times^C Y \subseteq S(\mu \cap ((X \cup Y) \times^C (X \cup Y)))$;

3. $X \times^C Y = S(\mu \cap ((X \cup Y) \times^C (X \cup Y)))$.