Proposition 18. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ then every neighborhood filter is a Cauchy filter, that it

$$\nu \sqsupseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\}$$

for every point x.

Proof. $\langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} = \langle (\mathsf{FCD})\nu \rangle \uparrow^{\mathrm{Ob}\,\nu} \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle \uparrow^{\mathrm{Ob}\,\nu} \{x\} = \nu \circ (\uparrow^{\mathrm{Ob}\,\nu} \{x\} \times^{\mathsf{RLD}} \uparrow^{\mathrm{Ob}\,\nu} \{x\}) \circ \nu^{-1} = \nu \circ \left(\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\nu; \mathrm{Ob}\,\nu)} \{(x; x)\} \right) \circ \nu^{-1} \sqsubseteq \nu \circ \mathrm{id}^{\mathsf{RLD}(\mathrm{Ob}\,\nu; \mathrm{Ob}\,\nu)} \circ \nu^{-1} = \nu \circ \nu^{-1} \sqsubseteq \nu$. □

Proposition 19. If a filter converges to a point, it is a low filter, provided that every neighborhood filter is a low filter.

Proof. Let $\mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\}$. Then $\mathcal{F} \times^{\mathsf{RLD}} \mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} \sqsubseteq \nu$. □

Corollary 20. If a filter converges to a point, it is a low filter, provided that $\nu \sqsupseteq \nu \circ \nu^{-1}$.

6 Maximal Cauchy filters

Lemma 21. Let S be a set of sets with $\prod \langle \uparrow^{\mathfrak{F}} \rangle S \neq 0^{\mathfrak{F}}$ (in other words, S has finite intersection property). Let $T = \{X \times X \mid X \in S\}$. Then

$$\bigcup T \circ \bigcup T = \bigcup S \times \bigcup S.$$

Proof. Let $x \in \bigcup S$. Then $x \in X$ for some $X \in S$. $\langle \bigcup T \rangle \{x\} \sqsupseteq \uparrow X \supseteq \bigcap S \neq \emptyset$. Thus

$\langle \bigcup T \circ \bigcup T \rangle \{x\} = \langle \bigcup T \rangle \langle \bigcup T \rangle \{x\} \in \langle \uparrow^{\mathsf{FCD}} \bigcup T \rangle \prod \langle \uparrow^{\mathfrak{F}} \rangle S \sqsupseteq \bigsqcup \{ \langle \uparrow^{\mathsf{FCD}} (X \times X) \rangle \prod \langle \uparrow^{\mathfrak{F}} \rangle S \mid X \in S\} = \bigsqcup \{\uparrow^{\mathfrak{F}} X \mid X \in S\} = \bigsqcup \langle \uparrow^{\mathfrak{F}} \rangle S$ that is $\langle \bigcup T \circ \bigcup T \rangle \{x\} \supseteq \bigcup S$. □

Corollary 22. Let S be a set of filters (on some fixed set) with nonempty meet. Let

$$T = \{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \mid \mathcal{X} \in S\}$$

Then

$$\bigsqcup T \circ \bigsqcup T = \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S.$$

Proof. $\bigsqcup T \circ \bigsqcup T = \prod \{\uparrow^{\mathfrak{F}} (X \circ X) \mid X \in \bigsqcup T\}$.

If $X \in \bigsqcup T$ then $X = \bigcup_{Q \in T} (P_Q \times P_Q)$ where $P_Q \in Q$. Therefore by the lemma we have

$$\bigcup \{P_Q \times P_Q \mid Q \in T\} \circ \bigcup \{P_Q \times P_Q \mid Q \in T\} = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q.$$

Thus $X \circ X = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q$.

Consequently $\bigsqcup T \circ \bigsqcup T = \prod \left\{ \uparrow^{\mathfrak{F}} \left(\bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q \right) \mid X \in \bigsqcup T \right\} \sqsupseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$.

$\bigsqcup T \circ \bigsqcup T \sqsubseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$ is obvious. □

Definition 23. I call an endoreloid ν *symmetrically transitive* iff for every symmetric endofuncoid $f \in \mathsf{FCD}(\mathrm{Ob}\,\nu; \mathrm{Ob}\,\nu)$ we have $f \sqsubseteq \nu \Rightarrow f \circ f \sqsubseteq \nu$.

Obvious 24. It is symmetrically transitive if at least one of the following holds:

1. $\nu \circ \nu \sqsubseteq \nu$;
2. $\nu \circ \nu^{-1} \sqsubseteq \nu$;
3. $\nu^{-1} \circ \nu \sqsubseteq \nu$.
4. $\nu^{-1} \circ \nu^{-1} \sqsubseteq \nu$.

Corollary 25. Every uniform space is symmetrically transitive.