Obvious 7. Every completely Cauchy space is a Cauchy space.

Proposition 8. $\bigsqcup^{\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}}S=\bigsqcup S$ for nonempty $S\in\mathscr{P}\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}$, provided that $\mathcal{F}$ is a fixed Cauchy filter on a completely Cauchy space.

Proof. $\mathcal{F}$ is proper. So for every nonempty $S\in\mathscr{P}\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}$ we have $\bigsqcap S\sqsupseteq\mathcal{F}\neq 0^{\mathfrak{F}(X)}$. Thus $\bigsqcup S$ is a Cauchy filter and so $\bigsqcup S\in\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}$. □

Proposition 9. If $\mathcal{F}$ is a fixed Cauchy filter on a completely Cauchy space, then the poset $\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}$ (with the induced order) is a complete lattice.

Proof. If $S\neq\emptyset$ then $\bigsqcup^{\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}}S=\bigsqcup S$. If $S=\emptyset$ then $\bigsqcup^{\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}}S=\mathcal{F}$. □

Corollary 10. If $\mathcal{F}$ is a fixed Cauchy filter on a completely Cauchy space, then the poset $\{\mathcal{X}\in\mathscr{C}\mid\mathcal{X}\sqsupseteq\mathcal{F}\}$ (with the induced order) has a maximum.

4 Relationships with symmetric reloids

Definition 11. Denote $(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})=\bigsqcup\{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\mid\mathcal{X}\in\mathscr{C}\}$.

Definition 12. $(\mathrm{Low})\nu$ (*low filters* for reloid ν) is a low filters space on U such that

$$\mathrm{PR}\,(\mathrm{Low})\nu=\{\mathcal{X}\in\mathfrak{F}^U\setminus\{0^{\mathfrak{F}}\}\mid\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq\nu\}.$$

Theorem 13. If $(U;\mathscr{C})$ is a low filters space, then $(U;\mathscr{C})=(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$.

Proof. If $\mathcal{X}\in\mathscr{C}$ then $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$ and thus $\mathcal{X}\in\mathrm{PR}\,(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$. Thus $(U;\mathscr{C})\sqsubseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$.

Let's prove $(U;\mathscr{C})\sqsupseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$.

Let $\mathcal{A}\in\mathrm{PR}\,(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$. We need to prove $\mathcal{A}\in\mathscr{C}$.

Really $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$. It is enough to prove that $\exists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

Suppose $\nexists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

For every $\mathcal{X}\in\mathscr{C}$ obtain $X_{\mathcal{X}}\in\mathcal{X}$ such that $X_{\mathcal{X}}\notin\mathcal{A}$ (if forall $X\in\mathcal{X}$ we have $X_{\mathcal{X}}\in\mathcal{A}$, then $\mathcal{X}\sqsupseteq\mathcal{A}$ what is contrary to our supposition).

It is now enough to prove $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\not\sqsubseteq\bigsqcup\{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}\mid\mathcal{X}\in\mathscr{C}\}$.

Really, $\bigsqcup\{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}}\mid\mathcal{X}\in\mathscr{C}\}=\uparrow^{\mathsf{RLD}(U;U)}\bigcup\{X_{\mathcal{X}}\times X_{\mathcal{X}}\mid\mathcal{X}\in\mathscr{C}\}$. So our claim takes the form $\bigcup\{X_{\mathcal{X}}\times X_{\mathcal{X}}\mid\mathcal{X}\in\mathscr{C}\}\notin\mathrm{GR}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A})$ that is $\forall A\in\mathcal{A}:\bigcup\{X_{\mathcal{X}}\times X_{\mathcal{X}}\mid\mathcal{X}\in\mathscr{C}\}\nsupseteq A\times A$ what is true because $X_{\mathcal{X}}\nsupseteq A$ for every $A\in\mathcal{A}$. □

Remark 14. The last theorem does not hold with $\mathcal{X}\times^{\mathsf{FCD}}\mathcal{X}$ instead of $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}$ (take $\mathscr{C}=\{\{x\}\mid x\in U\}$ for an infinite set U as a counter-example).

Remark 15. Not every symmetric reloid is in the form $(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$ for some Cauchy space $(U;\mathscr{C})$. The same Cauchy space can be induced by different uniform spaces. See http://math.stackexchange.com/questions/702182/different-uniform-spaces-having-the-same-set-of-cauchy-filters

[TODO: Is composition of two images of low filter spaces also a low filters space?]

5 More on Cauchy filters

Obvious 16. Low filter on an endoreloid ν is a filter $\mathcal{F}$ such that

$$\forall U\in\mathrm{GR}\,f\,\exists A\in\mathcal{F}:A\times A\subseteq U.$$

Remark 17. The above formula is the standard definition of Cauchy filters on uniform spaces.