Let G and H are graphs.

The exponential graph $\mathrm{MOR}(G;H)$ is defined by the formulas:

$\mathrm{Ob}\,\mathrm{MOR}(G;H) = (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}$;

$(f;g) \in \mathrm{GR}\,\mathrm{MOR}(G;H) \Leftrightarrow \forall (v;w) \in \mathrm{GR}\,G: (f(v);g(w)) \in \mathrm{GR}\,H$

for every $f, g \in \mathrm{Ob}\,\mathrm{MOR}(G;H) = (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}$.

Equivalently

$(f;g) \in \mathrm{GR}\,\mathrm{MOR}(G;H) \Leftrightarrow \forall (v;w) \in \mathrm{GR}\,G: g \circ \{(v;w)\} \circ f^{-1} \subseteq \mathrm{GR}\,H$

$(f;g) \in \mathrm{GR}\,\mathrm{MOR}(G;H) \Leftrightarrow g \circ (\mathrm{GR}\,G) \circ f^{-1} \subseteq \mathrm{GR}\,H$

$(f;g) \in \mathrm{GR}\,\mathrm{MOR}(G;H) \Leftrightarrow \langle f \times^{(C)} g \rangle \mathrm{GR}\,G \subseteq \mathrm{GR}\,H$

Evaluation $\varepsilon: (\mathrm{MOR}(G;H) \times G) \to H$

If $(f;g) \in \mathrm{GR}\,\mathrm{MOR}(G;H)$ and $x \in \mathrm{GR}\,G$ then $\varepsilon((f;g);x) = (fx;gx) = g \circ \{(x;x)\} \circ f^{-1}$

3.2 Exponentials in category Fcd

The below gives definitions for exponential object, (exponential) evaluation, and exponential transpose, but no proof is given that they are really exponential object, (exponential) evaluation, and exponential transpose. Please write `porton@narod.ru` if you find a proof.

If G, H are endofuncoids, then $\mathrm{MOR}(G;H)$ (*exponential object*) is an endofuncoid.

$$\mathrm{MOR}(G;H) = \bigsqcup \{t \in \operatorname{atoms} \mathsf{FCD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}; (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}) \mid (\mathrm{im}\,t) \circ G \circ (\mathrm{dom}\,t)^{-1} \sqsubseteq H\} = \bigsqcup \left\{t \in \operatorname{atoms} \mathsf{FCD}((\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}; (\mathrm{Ob}\,H)^{\mathrm{Ob}\,G}) \mid \langle (\mathrm{dom}\,t) \times^{(C)} (\mathrm{im}\,t) \rangle G \sqsubseteq H\right\}.$$

Evaluation:

$$\varepsilon((f \times^{(A)} g) \times^{(A)} x) = \langle f \rangle (\mathsf{RLD})_{\mathrm{in}} x \times^{\mathsf{FCD}} \langle g \rangle (\mathsf{RLD})_{\mathrm{in}} x.$$

$$\varepsilon((f \times^{(A)} g) \times^{(A)} x) = g \circ ((\mathsf{RLD})_{\mathrm{in}} x \times^{\mathsf{FCD}} (\mathsf{RLD})_{\mathrm{in}} x) \circ f^{-1} = \langle f \times^{(C)} g \rangle ((\mathsf{RLD})_{\mathrm{in}} x \times^{\mathsf{FCD}} (\mathsf{RLD})_{\mathrm{in}} x)$$

for atomic $f, g \in \mathsf{FCD}(\mathrm{Ob}\,G; \mathrm{Ob}\,H)$.

Evaluation $\varepsilon: \mathrm{MOR}(A;B) \times A \to B$

$$\varepsilon(F \times x) = \bigsqcup \left\{g \circ (x \times^{\mathsf{FCD}} x) \circ f^{-1} \mid f, g \in \operatorname{atoms}^{\mathsf{FCD}(\mathrm{Ob}\,A; \mathrm{Ob}\,B)}, f \times g \not\asymp F\right\}.$$

$\varepsilon(F \times x) = \bigsqcup \{g_1 \times^{\mathsf{FCD}} f_1 \mid f_0, g_0, f_1, g_1 \in \operatorname{atoms}^{\mathfrak{F}}, (f_0 \times^{\mathsf{FCD}} f_1) \times (g_0 \times^{\mathsf{FCD}} g_1) \not\asymp F\}$.

Proposition 15. *$\varepsilon: \mathrm{MOR}(A;B) \times A \to B$.*

Proof. We need to prove $\varepsilon \circ (\mathrm{MOR}(A;B) \times A) \sqsubseteq B \circ \varepsilon$. ?? □

Transpose $\sim f: Z \to \mathrm{MOR}(A;B)$ for a morphism $f: Z \times A \to B$

$$(\sim f)x = \bigsqcup \{b \times^{\mathsf{FCD}} \langle f \rangle (x \times b) \mid b \in \operatorname{atoms}^{\mathsf{FCD}}\}.$$

Proposition 16. *$\sim f: Z \to \mathrm{MOR}(A;B)$.*

Proof. We need to prove $\sim f \circ Z \sqsubseteq \mathrm{MOR}(A;B) \circ \sim f$ whenever $f \circ (Z \times A) \sqsubseteq B \circ f$. ?? □

Awoday 6.5 "Equational definition" gives a simple way to check cartesian closed categories.

It's enough to prove:

1. $\varepsilon \circ (\sim f \times 1_A) = f$
2. $\varepsilon \circ \sim (g \times 1_A) = g$

$\varepsilon(\sim f \times 1_A)x = \varepsilon((\sim f)x \times x) = \bigsqcup \{g_1 \times^{\mathsf{FCD}} f_1 \mid f_0, g_0, f_1, g_1 \in \operatorname{atoms}^{\mathfrak{F}}, (f_0 \times^{\mathsf{FCD}} f_1) \times (g_0 \times^{\mathsf{FCD}} g_1) \not\asymp (\sim f)x\} = ??$

??