Proof. Let $H \in \mathrm{up}(g \circ f)$. Then $\exists F \in \mathrm{up}\, f: H \sqsupseteq g \circ F$ that is $H \in \mathrm{up}(g \circ F)$. Thus $\exists G \in \mathrm{up}\, g$ such that $H \sqsupseteq G \circ F$. □

3 Join of transitive reloids

$(f \sqcup g) \circ (f \sqcup g) \sqsupseteq (f \circ f) \sqcup (g \circ g)$ [need other direction]

Join of compositions of all finite sequences of f and g (in any order). It is equivalent to taking all alternating $S^*(f \circ g \circ f \circ \ldots \circ f)$ starting or ending with f or g.

"Relationships between completeness properties" in https://en.wikipedia.org/wiki/Completeness_%28order_theory%29

or alternatively:

$\mu = (f \sqcup g) \sqcup ((f \sqcup g) \circ (f \sqcup g)) \sqcup ((f \sqcup g) \circ (f \sqcup g) \circ (f \sqcup g)) \sqcup \ldots$

$\mu \circ \mu \sqsupseteq \mu$

"No similarly useful description of a subbase for the infimum of a family of quasi-uniformities is known." by http://www.sciencedirect.com/science/article/pii/S0166864107000181

—

$\mathrm{up}\left(\bigsqcup_{n \in \mathbb{N}, f_{n,i} \in S} (f_{n,n} \circ \ldots \circ f_{n,0})\right) = \bigcap_{n \in \mathbb{N}, f_{n,i} \in S, F_{n,i} \in \mathrm{up}\, f_{n,i}} (F_{n,n} \circ \ldots \circ F_{n,0})$

Lemma 14. *$\mu = \bigsqcup_{n \in \mathbb{N}, f_{n,i} \in S} (f_{n,n} \circ \ldots \circ f_{n,0})$ is a transitive reloid, for every set S of endoreloids (on the same set).*

Proof. Denote $[U_n: n \in \mathbb{N}] = \bigsqcup_{n \in \mathbb{N}} (U_n \circ \ldots \circ U_0)$.

Let $U_{n,X} \in \mathrm{up}\, X$ for all $n \in \mathbb{N}$ and $X \in S$.

$\left[\bigcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right]$ is ??

We need to prove $\mu \circ \mu \sqsubseteq \mu$.

$\left[\bigsqcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right] \in \mathrm{up}\, \mu$.

Let $U_{n,X} \in X$ and $U_n = \bigcup_{X \in S} U_{n,X}$.

Then $\left[\bigsqcup_{X \in S} U_{2n,X}: n \in \mathbb{N}\right] \circ \left[\bigsqcup_{X \in S} U_{2n-1,X}: n \in \mathbb{N}\right] \subseteq \left[\bigsqcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right]$ because?? $U_{2n,X} \circ U_{2n-1,X} \in \left[\bigsqcup_{X \in S} U_{n,X}: n \in \mathbb{N}\right]$?? [TODO: No need for reindexation.]

??

U_n is the join of all compositions of n-tuples. They form a generalized filter base. Thus it's enough to show that every U_n can be decomposed into smaller n-tuples. But that's obvious. (It isn't because it is an infinite join!)

Thus is above $U'_n \circ U''_n$.

Thus μ is also decomposed, because every its element is minorated as shown above.

??

Take $P \in \mathrm{up}\, \mu$. Then $P \in \mathrm{up}\, A$ for every $A \in \mathrm{up}(f_{n,n} \circ \ldots \circ f_{n,0})$.

$P \sqsupseteq F_{N,N} \circ \ldots \circ F_{N,0}$ for some $F_{N,i} \in \mathrm{up}\, f_{N,i}$ for all $N \in \mathbb{N}$.

Thus $P \sqsupseteq \bigsqcup_{n \in \mathbb{N}, f_i \in S} (F_{N,N} \circ \ldots \circ F_{N,0})$ where $F_{N,i} \in \mathrm{up}\, f_{N,i}$.

Take $F'_N = F_{N,\lfloor N/2 \rfloor} \circ \ldots \circ F_{N,0}$ and $F''_N = F_{N,N} \circ \ldots \circ F_{N,\lfloor N/2 \rfloor + 1}$

This way we exhaust all possible values?

$(F_{n,n} \circ \ldots \circ F_{n,n}) \circ (F_{m,m} \circ \ldots \circ F_{m,0}) \sqsubseteq P$. But this is obvious.

Thus easily follows $P \sqsupseteq \left(\bigsqcup_{n \in \mathbb{N}, f_i \in S} (F_{n,n} \circ \ldots \circ F_{n,0})\right) \circ \left(\bigsqcup_{n \in \mathbb{N}, f_i \in S} (F_{n,n} \circ \ldots \circ F_{n,0})\right)$; $P \in \mathrm{up}(\mu \circ \mu)$. □

Alternative formula: $\bigcup \langle \mathrm{GR} \rangle^* S \sqcup Z(\bigcup \langle \mathrm{GR} \rangle^* S) \sqcup Z(Z(\bigcup \langle \mathrm{GR} \rangle^* S)) \sqcup \ldots$

[TODO: Both for reloids and for Cauchy spaces Γ_i as in the attached article in email.]

[TODO: Also for funcoids (using (FCD)).]

3.1 Exponentials in category of graphs

http://arxiv.org/pdf/math/0605275.pdf definition 2.3 defines exponential graph