Represent $H$ as a meet of elements of $\Gamma$ ($H = \bigsqcap \mathrm{up}^{\Gamma} H$). For each $H' \in \mathrm{up}^{\Gamma} H$ choose $F_{H'} \in \mathrm{up}^{\Gamma} f$, $G_{H'} \in \mathrm{up}^{\Gamma} g$ such that $H' \sqsupseteq G_{H'} \circ F_{H'}$. Moreover we can choose maximal $F_{H'}$, $G_{H'}$ such that this inequatly holds. Then take $F = \bigsqcap_{H' \in \mathrm{up}^{\Gamma} H} F_{H'}$ and $G = \bigsqcap_{H' \in \mathrm{up}^{\Gamma} H} G_{H'}$. [FIXME: Does $F \in \mathrm{up}\, f$?] [TODO: If this does not work, then seems that there is a counter-example, because it is the stongest.]

[TODO: We MUST take maximal rather than arbitrary $F_{H'}, G_{H'}$. Otherwise take $f = 1$, $g = \mathrm{id}_{\Omega}$. Then if we take $F_{H'} = 1$ and replace all possible $G_{H'} \rightarrow G_{H'} \setminus \{(a; b)\}$, then $G = \bot \notin \mathrm{up}\, g$]

??

$H \in \mathrm{up}(g \circ f) \Rightarrow \langle H \rangle^{\ast} X \sqsupseteq \langle g \circ f \rangle^{\ast} X = \langle g \rangle \langle f \rangle^{\ast} X$.

For every $X$ take $Y_X \in \mathrm{up}\, \langle f \rangle^{\ast} X$ such that $\langle g \rangle Y_X \sqsubseteq \langle H \rangle^{\ast} X$. If $g$ is complete, we may assume that $Y_X$ is a maximal set for which $\langle g \rangle Y_X \sqsubseteq \langle H \rangle^{\ast} X$ holds.

Take $F = \bigsqcap \{(X \times Y_X) \sqcup (\overline{X} \times \top) \mid X \in \mathscr{T} \mathrm{Src}\, f\}$. But is $F$ complete or co-complete, or if meet is taken on Rel, do we have $F \in \mathrm{up}\, f$?

??

Let $g$ be principal. Let $a$ be an atomic filter. Take (??not always possible) $Y_a \in \mathrm{up}\, \langle f \rangle^{\ast} a$ such that $\langle g \rangle Y_a \sqsubseteq \langle H \rangle a$

Let $F = \bigsqcup_{a \in \mathrm{atoms}} (a \times Y_a)$ - also co-complete

??

Let for each $b \in \mathrm{atoms}\, \langle f \rangle^{\ast} a$ define $Z_b = ??$

??

Let $f$ be complete. Replace it with principal funcoid $F$, such that $\langle f \rangle^{\ast} \{x\} \sqsubseteq \langle F \rangle^{\ast} x \sqsubseteq Y_{\{x\}}$. Prove $g \circ F \sqsubseteq H$

??

Split $F$ into a join of monovalued functions. This does not work because every function produces its own $g$.

??

$G \circ f \sqsubseteq H$; $G \circ \mathrm{Compl}\, f = \mathrm{Compl}(G \circ f) \sqsubseteq H$

??

$\langle G \circ f \rangle^{\ast} \mathcal{X} = \langle G \rangle \langle f \rangle^{\ast} \mathcal{X} = \langle G \rangle \bigsqcap \langle \langle f \rangle^{\ast} \rangle^{\ast} \mathrm{up}\, \mathcal{X} \sqsubseteq \bigsqcap \langle \langle G \circ f \rangle^{\ast} \rangle^{\ast} \mathrm{up}\, \mathcal{X} \sqsubseteq ?? \sqsubseteq \langle g \circ f \rangle^{\ast} \mathcal{X} \sqsubseteq \langle H \rangle \mathcal{X}$

??

Find maximal [FIXME: there is no maximal because composition is not distributive over arbitrary joins.] funcoids $F$ and $G$ such that $G \circ F \sqsubseteq H$, then prove they are principal (or (co-)complete)

??

Replace $G$ with $G^M$ mapping $x \mapsto \langle G \rangle^{\ast} \{x\}$, Then ?? consider it as an isomorphism between sets.

$(Q \circ P)^M x = \langle Q \circ P \rangle^{\ast} \{x\} = \langle Q \rangle \langle P \rangle^{\ast} \{x\} = \langle Q \rangle P^M x$

$g \circ F \sqsubseteq H \Leftrightarrow \langle g \rangle \circ F^M \sqsubseteq H^M$

??

$f = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f$, $g = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g$; $H \in \mathrm{up}(g \circ f)$

??

Use Todd Trimble's idea with $\xi$: $H \in \mathrm{up}(g \circ f) \Leftrightarrow (\mathcal{A} H^{\circledast} \mathcal{C} \Leftarrow \mathcal{A} (g \circ f)^{\circledast} \mathcal{C}) \Leftrightarrow (\mathcal{A} H^{\circledast} \mathcal{C} \Leftarrow \exists \mathcal{B}: (\mathcal{A} f^{\circledast} \mathcal{B} \wedge \mathcal{B} g^{\circledast} \mathcal{C})) \Leftrightarrow (A H^{\circledast} C \Leftarrow \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B))$ [FIXME: check direction of implication]

$\{(A; C) \mid C \sqsupseteq \langle H \rangle A\} \supseteq \{(A; C) \mid \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)\}$.

Suppose $A \in \mathrm{dom}\, \{(A; C) \mid \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)\}$ or $C \in \mathrm{im}\, \{(A; C) \mid \exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)\}$.

Then $\exists B: (B \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B)$. Take $B_{A,C}$ such that $B_{A,C} \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B_{A,C}$

Take $B'_A = \bigcap_{C \in ??} B_{A,C}$.

Then $B'_A \sqsupseteq \langle f \rangle A \wedge C \sqsupseteq \langle g \rangle B'_A$. [FIXME: it does not hold, only $B'_A \sqsupseteq \mathrm{Cor}\, \langle f \rangle A$]

Take co-complete funcoid $\langle F \rangle A = B'_A$. It is possible?? because $B'_{X \sqcup Y} = \bigcap_{C \in ??} B_{X \sqcup Y, C} = ??$

$B_{A, X \sqcup Y} \sqsupseteq \langle f \rangle (X \sqcup Y) \wedge C \sqsupseteq \langle g \rangle B_{X \sqcup Y, C}$

??

Instead of intersecting funcoids, consider join $f = \bigsqcup_{X \in ??} X \times Y_X$ or $f = \bigsqcup_{\mathcal{X} \in ??} \mathcal{X} \times Y_{\mathcal{X}}$. It is enough to consider ultrafilters $\mathcal{X}$. □

**Theorem 13.** *$\forall H \in \mathrm{up}(g \circ f) \exists F \in \mathrm{up}\, f, G \in \mathrm{up}\, g: H \sqsupseteq G \circ F$ for every composable funcoids $f$ and $g$. [TODO: Also state it for reloids.]*