Proof. For counterexample take $\mu = \top \setminus 1$.

$\langle\mu\rangle\{x\} = \top \setminus \{x\}$ (thus $\mu \not\sqsupseteq 1^{\mathsf{FCD}}$) and $\langle\mu\rangle a = \top$ for a nontrivial ultrafilter a.

Let $\mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob}\mu) \setminus \{\bot\}$ and $\mathcal{X} \sqcup \mathcal{Y} = \top$. If $\mathcal{X}$ is a trivial ultrafilter then $\langle\mu\rangle\mathcal{X} = \top \setminus \{x\}$ adn thus $\langle\mu\rangle\mathcal{X} \not\asymp \mathcal{Y}$, otherwise $\langle\mu\rangle\mathcal{X} \not\asymp \mathcal{Y}$. So in any case $\mathcal{X}\ [\mu]\ \mathcal{Y}$. Funcoid μ is connected. □

Proposition 2409. If there is a nonzero-length path regarding μ in the filter $\mathcal{A}$ between any two its atomic subfilters, then it is connected regarding μ.

Proof. Let $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A}$, $\mathcal{X} \neq \bot$, $\mathcal{Y} \neq \bot$. Let $p_0, \ldots, p_n$ $(n \geq 1)$ be a path in $\mathcal{A}$ and $p_0 \in \operatorname{atoms}\mathcal{X}$ and $p_n \in \operatorname{atoms}\mathcal{Y}$. Then (take $k = \min\{ i \in \{0, \ldots, n-1\} \mid p_{i+1} \in \operatorname{atoms}\mathcal{Y}\}$) there are p_k, p_{k+1} such that $p_k \in \operatorname{atoms}\mathcal{X}$, $p_{k+1} \in \operatorname{atoms}\mathcal{Y}$. But $p_k\ [\mu]\ p_{k+1}$ by definition of path. Thus $\mathcal{X}\ [\mu]\ \mathcal{Y}$. □

Proposition 2410. If a filter $\mathcal{A}$ is connected regarding funcoid μ reflexive on $\mathcal{A}$ then it is connected regarding every μ^n for $n \in \mathbb{Z}_+$.

Proof. Let $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A}$, $\mathcal{X} \neq \bot$, $\mathcal{Y} \neq \bot$. We have $\langle\mu\rangle\mathcal{X} \not\asymp \mathcal{Y}$.

Then $\langle\mu\rangle\mathcal{X} \not\sqsubseteq \mathcal{X}$; therefore by reflexivity $\langle\mu\rangle\mathcal{X} \sqsupset \mathcal{X}$. Repeating this step we get $\langle\mu\rangle\langle\mu\rangle\mathcal{X} \sqsupset \mathcal{X}$ that is $\langle\mu^2\rangle\mathcal{X} \sqsupset \mathcal{X}$, etc.

We have $\langle\mu^n\rangle\mathcal{X} \sqsupset \mathcal{X}$ and thus $\langle\mu^n\rangle\mathcal{X} \not\asymp \mathcal{Y}$ that is $\mathcal{X}\ [\mu^n]\ \mathcal{Y}$. □

Example 2411. Connected funcoid without a path between given ultrafilters.

Proof. Consider $|\mathbb{R}|$. It is connected (prove!) but there is no path (prove!) between two distinct singletons. □

Theorem 2412. If meet of two connected (regarding a funcoid) filters is nonleast, then their join is connected.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be intersecting filters, both connected regarding an endofuncoid μ. Let $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A} \sqcup \mathcal{B}$ for proper filters $\mathcal{X}$, $\mathcal{Y}$. Then either $\mathcal{X}$ or $\mathcal{Y}$ intersects $\mathcal{A} \sqcap \mathcal{B}$. Without loss of generality assume $\mathcal{X} \sqcap \mathcal{A} \sqcap \mathcal{B} \neq \bot$. Also $\mathcal{Y}$ intersects either $\mathcal{A}$ or $\mathcal{B}$. Without loss of generality assume $\mathcal{Y} \sqcap \mathcal{A} \neq \bot$.

Note $\mathcal{X} \sqcap \mathcal{A} \neq \bot$.

We have $(\mathcal{X} \sqcap \mathcal{A}) \sqcup (\mathcal{Y} \sqcap \mathcal{A}) = (\mathcal{X} \sqcup \mathcal{Y}) \sqcap \mathcal{A} = (\mathcal{A} \sqcup \mathcal{B}) \sqcap \mathcal{A} = \mathcal{A}$. So $\mathcal{X} \sqcap \mathcal{A}\ [\mu]\ \mathcal{Y} \sqcap \mathcal{A}$ because $\mathcal{A}$ is connected, consequently $\mathcal{X}\ [\mu]\ \mathcal{Y}$ that is $\mathcal{A} \sqcup \mathcal{B}$ is connected. □

Theorem 2413. If meet of two connected (regarding a reloid) filters is nonempty, then their join is connected.

Proof. Let $S_1^*(\mu \sqcap (\mathcal{A} \times \mathcal{A})) = \mathcal{A} \times \mathcal{A}$; $S_1^*(\mu \sqcap (\mathcal{B} \times \mathcal{B})) = \mathcal{B} \times \mathcal{B}$ for filters $\mathcal{A} \not\asymp \mathcal{B}$.

$S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) = S_1^*(\mu \sqcap ((\mathcal{A} \times \mathcal{A}) \sqcup (\mathcal{B} \times \mathcal{B}) \sqcup (\mathcal{A} \times \mathcal{B}) \sqcup (\mathcal{B} \times \mathcal{A}))) \sqsupseteq S_1^*(\mu \sqcap (\mathcal{A} \times \mathcal{A})) \sqcup S_1^*(\mu \sqcap (\mathcal{B} \times \mathcal{B})) \sqsupseteq (\mathcal{A} \times \mathcal{A}) \sqcup (\mathcal{B} \times \mathcal{B})$.

Let for example $x \in \operatorname{atoms}\mathcal{A}$. Then $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B})))\rangle x \sqsupseteq \mathcal{A}$ and (taking into account $\mathcal{A} \not\asymp \mathcal{B}$):

$$\langle\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))\rangle\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B})))\rangle x \sqsupseteq \mathcal{B}.$$

Thus $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B})))\rangle x \sqsupseteq \mathcal{A}$ and $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B})))\rangle x \sqsupseteq \mathcal{B}$ for every ultrafilter $x \in \operatorname{atoms}(\mathcal{A} \sqcup \mathcal{B})$, that is $\langle S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B})))\rangle x \sqsupseteq \mathcal{A} \sqcup \mathcal{B}$. So $S_1^*(\mu \sqcap ((\mathcal{A} \sqcup \mathcal{B}) \times (\mathcal{A} \sqcup \mathcal{B}))) \sqsupseteq \mathcal{A} \sqcup \mathcal{B}$ that is $\mathcal{A} \sqcup \mathcal{B}$ is connected. □

Corollary 2414. Distinct connected components (for both a funcoid or a reloid) don't intersect.

Proof. If connected components $\mathcal{A} \neq \mathcal{B}$ intersect, then $\mathcal{A} \sqcup \mathcal{B}$ is a connected filter and either $\mathcal{A} \sqcup \mathcal{B} \sqsupset \mathcal{A}$ or $\mathcal{A} \sqcup \mathcal{B} \sqsupset \mathcal{B}$ what contradicts to the definition of connected components. □