CHAPTER 14

More on connectedness

1. For topological spaces

PROPOSITION 2395. The following are pairwise equivalent:

1°. a topological space on a set U is connected. FiXme: definition; can the topological definition be generalized to filters?

2°. U is connected regarding $f \sqcup f^{-1}$ if f is the corresponding complete funcoid.

3°. U is connected regarding $f \sqcup f^{-1}$ if f is the corresponding closure space.

4°. U is connected regarding $f \circ f^{-1}$ if f is the corresponding complete funcoid.

PROOF. ?? □

PROPOSITION 2396. There are filters $\mathcal{A}$, $\mathcal{B}$, such that there are no filters $\mathcal{X} \sqsubseteq \mathcal{A}$, $\mathcal{Y} \sqsubseteq \mathcal{B}$ such that $\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A} \sqcup \mathcal{B}$ and $\mathcal{X} \asymp \mathcal{Y}$.

PROOF. https://math.stackexchange.com/questions/2639206

(It also follows that sometimes $Z(Da)$ is not a complete lattice, because otherwise we could prove this theorem.) □

PROPOSITION 2397. If $\mathcal{A}$, $\mathcal{B}$ are filters and $\mathcal{A} \sqcup \mathcal{B} = U$ is principal filter, then there are sets $X \sqsubseteq \mathcal{A}$, $Y \sqsubseteq \mathcal{B}$ such that $X \sqcup Y = U$ and $X \asymp Y$.

PROOF. Take $X = \operatorname{Cor} \mathcal{A}$ and $Y' = \operatorname{Cor} \mathcal{B}$. Then $X \sqcup Y' = U$ because of co-separability of $\mathfrak{F}(U)$. Take $Y = U \setminus X$. Then $X \sqcup Y = U$ and $X \asymp Y$. □

PROPOSITION 2398. A principal filter A is connected regarding endofuncoid μ iff

$$\forall X, Y \in \mathscr{P}(\operatorname{Ob} \mu) \setminus \{\bot\} : (X \sqcup Y = A \wedge X \asymp Y \Rightarrow X \ [\mu] \ Y).$$

PROOF. Easily follows from ??. □

DEFINITION 2399. *Connected component* of a filter regarding a funcoid or a reloid is a maximal connected subfilter of this filter.

OBVIOUS 2400. Subfilter of a connected filter is connected.

PROPOSITION 2401. If U is a principal filter, then it is connected regarding μ iff it is connected regarding $S(\mu)$. FiXme: It should be presented as a corollary of a below theorem.

PROOF. If U is connected regarding μ, it is connected regarding $S(\mu)$, obviously.

Suppose U is connected regarding $S(\mu)$. Then for $X, Y \in \mathscr{P}(\operatorname{Ob} \mu) \setminus \{\bot\}$ we have if $X \sqcup Y = U$ and $X \asymp Y$, then $X \ [S(\mu)] \ Y$. So $X \times Y \not\asymp 1 \sqcup \mu \sqcup \mu^2 \sqcup \ldots$ and thus by distributivity for principal filter we have $X \times Y \not\asymp \mu^n$ for some $n \geq$?? that is $X \ [\mu^n] \ Y$ for some n and thus there are atomic filters $p_0, \ldots, p_n$ such that $p_0 \in \operatorname{atoms}^{\mathfrak{F}} X$, $p_n \in \operatorname{atoms}^{\mathfrak{F}} Y$ and $p_i \ [\mu] \ p_{i+1}$. Thus there is k such that $p_k \ [\mu] \ p_{k+1}$ and $p_k \in \operatorname{atoms}^{\mathfrak{F}} X$, $p_{k+1} \in \operatorname{atoms}^{\mathfrak{F}} Y$. Thus $X \ [\mu] \ Y$. We have U connected regarding μ. □