Lemma 2390. (assuming conjecture 1444) For every $U \in \operatorname{up}\mu$ (where μ is a T_4 topological space) such that $\neg\left(A \left[U \circ U^{-1}\right]^* B\right)$ there is $W \in \operatorname{up}\mu$ such that $U \circ U^{-1} \sqsupseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$. For it holds $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$. We can assume that $\langle W\rangle^* X$ is open for every set X.

Proof. Applying the previous lemma twice, we have some open $W \in \operatorname{up}\mu$ such that

$$U \circ U^{-1} \sqsupseteq W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}$$

and $\neg\left(A \left[W \circ W^{-1}\right]^* B\right)$. From this easily follows that

$$U \circ U^{-1} \sqsupseteq \mu^{-1} \circ W \circ W^{-1} \circ W \circ W^{-1}.$$

□

A modified proof of Urysohn's lemma follows. This proof is in part based on [1]. (I attempt to find common generalization of Urysohn's lemma and results from [1]).

$\mathbb{Q}_2 \overset{\text{def}}{=} \left\{ \frac{k/2^n}{k,n \in \mathbb{N}, 0<k<2^n} \right\}$.

Theorem 2391. Urysohn's lemma (see Wikipedia) for disjoint closed sets A and B and function f on a topological space μ (considered as complete funcoid).

Proof. (assuming conjecture 1444) (used ProofWiki among other sources)

Because A and B are disjoint closed sets, we have $\langle\mu\rangle^* A \asymp \langle\mu\rangle^* B$. Thus by the corollary 2388 take $S_0 \in \operatorname{up}\mu$ and $\neg\left(A \left[S_0 \circ S_0^{-1}\right]^* B\right)$.

We have $\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1} \sqsubseteq \mu \circ \mu^{-1}$ that is $\operatorname{up}(\mu \circ \mu^{-1} \circ \mu \circ \mu^{-1}) \supseteq \operatorname{up}(\mu \circ \mu^{-1})$.

Let's prove by induction: There is a sequence S of binary relations starting with S_0 such that $\neg\left(A \left[S_i \circ S_i^{-1}\right]^* B\right)$ and $S_i \circ S_i^{-1} \sqsupseteq \mu^{-1} \circ S_{i+1} \circ S_{i+1}^{-1} \circ S_{i+1} \circ S_{i+1}^{-1}$. It directly follows from the lemma (and uses the conjecture).

Denote $U_i = S_{i+1} \circ S_{i+1}^{-1}$. We have $U_i \sqsupseteq \mu^{-1} \circ U_{i+1} \circ U_{i+1}$ and $\neg\left(A \left[U_i\right]^* B\right)$.

By reflexivity of μ we have $U_{i+1} \subseteq U_{i+1} \circ U_{i+1} \subseteq U_i$.

Define fractional degree of U: $U^r \overset{\text{def}}{=} U_1^{r_1} \circ \ldots \circ U_{l_r}^{r_{l_r}}$ for every $r \in \mathbb{Q}_2$ where $r_1 \ldots r_{l_r}$ is the binary expansion of r.

Prove $U_r \subseteq U_0$. It is enough to prove $U_0 \supseteq U_1 \circ \ldots \circ U_{l_r}$. It follows from $U_2 \circ \ldots \circ U_{l_r} \subseteq U_1$, $U_3 \circ \ldots \circ U_{l_r} \subseteq U_2$, …, $U_{l_r} \subseteq U_{l_r - 1}$ what was shown above.

Let's prove: For each $p, q \in \mathbb{Q}_2$ such that $p < q$ we have $\mu^{-1} \circ U^p \sqsubseteq U^q$. We can assume binary expansion of p and q be the same length c (add zeros at the end of the shorter one). Now it is enough to prove

$$U_k \circ U_{k+1}^{q_{k+1}} \circ \cdots \circ U_c^{q_c} \sqsupseteq \mu^{-1} \circ U_{k+1}^{p_{k+1}} \circ U_{k+2}^{p_{k+2}} \circ \cdots \circ U_c^{p_c}.$$

But for this it's enough

$$U_k \sqsupseteq \mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c$$

what can be easily proved by induction: If $k = c$ then it takes the form $U_k \sqsupseteq \mu^{-1}$ what is obvious. Suppose it holds for k. Then $U_{k-1} \sqsupseteq \mu^{-1} \circ U_k \circ U_k \sqsupseteq \mu^{-1} \circ U_k \circ \mu^{-1} \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c \sqsupseteq \mu^{-1} \circ U_k \circ U_{k+1} \circ U_{k+2} \circ \cdots \circ U_c$, that is it holds for all natural $k \leq c$.

It is easy to prove that $\langle U^r\rangle^* X$ is open for every set X.

We have $\left\langle\mu^{-1}\right\rangle^* \langle U^p\rangle^* X \sqsubseteq \langle U^q\rangle^* X$.

$$f(z) \overset{\text{def}}{=} \inf\left(\{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q\rangle^* A} \right\}\right).$$

f is properly defined because $\{1\} \cup \left\{ \frac{q \in \mathbb{Q}_2}{z \in \langle U^q\rangle^* A} \right\}$ is nonempty and bounded.