## 9. Maximal Cauchy filters

LEMMA 2356. Let $S$ be a set of sets with $\prod\langle\uparrow^{\mathfrak{F}}\rangle^* S \neq 0^{\mathfrak{F}}$ (in other words, $S$ has finite intersection property). Let $T = \left\{ \frac{X\times X}{X\in S} \right\}$. Then

$$\bigcup T \circ \bigcup T = \bigcup S \times \bigcup S.$$

PROOF. Let $x \in \bigcup S$. Then $x \in X$ for some $X \in S$. $\langle\bigcup T\rangle\{x\} \sqsupseteq\uparrow X \supseteq \bigcap S \neq \emptyset$. Thus
$\langle\bigcup T \circ \bigcup T\rangle\{x\} = \langle\bigcup T\rangle\langle\bigcup T\rangle\{x\} \in \left\langle\uparrow^{\mathsf{FCD}} \bigcup T\right\rangle \prod\langle\uparrow^{\mathfrak{F}}\rangle S \sqsupseteq \bigsqcup\left\{ \frac{\langle\uparrow^{\mathsf{FCD}}(X\times X)\rangle\prod\langle\uparrow^{\mathfrak{F}}\rangle S}{X\in S} \right\} = \bigsqcup\left\{ \frac{\uparrow^{\mathfrak{F}} X}{X\in S} \right\} = \bigsqcup\langle\uparrow^{\mathfrak{F}}\rangle S$ that is $\langle\bigcup T \circ \bigcup T\rangle\{x\} \supseteq \bigcup S$. □

COROLLARY 2357. Let $S$ be a set of filters (on some fixed set) with nonempty meet. Let

$$T = \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in S} \right\}$$

Then

$$\bigsqcup T \circ \bigsqcup T = \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S.$$

PROOF. $\bigsqcup T \circ \bigsqcup T = \prod\left\{ \frac{\uparrow^{\mathfrak{F}}(X\circ X)}{X\in\bigsqcup T} \right\}$.

If $X \in \bigsqcup T$ then $X = \bigcup_{Q\in T}(P_Q \times P_Q)$ where $P_Q \in Q$. Therefore by the lemma we have

$$\bigcup\left\{ \frac{P_Q \times P_Q}{Q \in T} \right\} \circ \bigcup\left\{ \frac{P_Q \times P_Q}{Q \in T} \right\} = \bigcup_{Q\in T} P_Q \times \bigcup_{Q\in T} P_Q.$$

Thus $X \circ X = \bigcup_{Q\in T} P_Q \times \bigcup_{Q\in T} P_Q$.

Consequently $\bigsqcup T \circ \bigsqcup T = \prod\left\{ \frac{\uparrow^{\mathfrak{F}}\left(\bigcup_{Q\in T} P_Q\times\bigcup_{Q\in T} P_Q\right)}{X\in\bigsqcup T} \right\} \sqsupseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$.

$\bigsqcup T \circ \bigsqcup T \sqsubseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$ is obvious. □

DEFINITION 2358. I call an endoreloid $\nu$ *symmetrically transitive* iff for every symmetric endofuncoid $f \in \mathsf{FCD}(\operatorname{Ob}\nu, \operatorname{Ob}\nu)$ we have $f \sqsubseteq \nu \Rightarrow f \circ f \sqsubseteq \nu$.

OBVIOUS 2359. It is symmetrically transitive if at least one of the following holds:

1°. $\nu \circ \nu \sqsubseteq \nu$;
2°. $\nu \circ \nu^{-1} \sqsubseteq \nu$;
3°. $\nu^{-1} \circ \nu \sqsubseteq \nu$.
4°. $\nu^{-1} \circ \nu^{-1} \sqsubseteq \nu$.

COROLLARY 2360. Every uniform space is symmetrically transitive.

PROPOSITION 2361. $(\mathrm{Low})\nu$ is a completely Cauchy space for every symmetrically transitive endoreloid $\nu$.

PROOF. Suppose $S \in \mathscr{P}\left\{ \frac{\mathcal{X}\in\mathfrak{F}}{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq\nu} \right\}$.

$\bigsqcup\left\{ \frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in S} \right\} \sqsubseteq \nu$; $\bigsqcup\left\{ \frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in S} \right\} \circ \bigsqcup\left\{ \frac{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}}{\mathcal{X}\in S} \right\} \sqsubseteq \nu$; $\bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S \sqsubseteq \nu$ (taken into account that $S$ has nonempty meet). Thus $\bigsqcup S$ is Cauchy. □

PROPOSITION 2362. The neighbourhood filter $\langle(\mathsf{FCD})\nu\rangle^*\{x\}$ of a point $x \in \operatorname{Ob}\nu$ is a maximal Cauchy filter, if it is a Cauchy filter and $\nu$ is a reflexive reloid. FiXme: Does it holds for all low filters?